Homework 5

due on Monday, 16 March 2009 by 12:50 PM

1. Find the length L of part of the curve $y = \ln x$ for $x \in [1, e]$.

$$L = \int_1^e \sqrt{1+(y')^2}\,dx = \int_1^e \sqrt{1+x^{-2}}\,dx = \int_1^e \frac{\sqrt{x^2+1}}{x}\,dx$$

Let $x = \tan\alpha$. Then we have

$$\begin{aligned}
\int \frac{\sqrt{x^2+1}}{x}\,dx &= \int \frac{\sqrt{\tan^2\alpha+1}}{\tan\alpha}\overbrace{\sec^2\alpha\,d\alpha}^{dx} = \int \frac{\sec\alpha\,\overbrace{(\tan^2\alpha+1)}^{\sec^2\alpha}}{\tan\alpha}\,d\alpha \\
&= \int \tan\alpha\sec\alpha\,d\alpha + \int \frac{\sec\alpha}{\tan\alpha}\,d\alpha = \sec\alpha + \int \csc\alpha\,d\alpha \\
&= \sec\alpha + \ln|\csc\alpha - \cot\alpha| + C = \sqrt{x^2+1} + \ln\left|\frac{\sqrt{x^2+1}-1}{x}\right| + C
\end{aligned}$$

2. Evaluate $\displaystyle\int \frac{1}{\cos x(1-\sin x)^2}\,dx$.
Let $u = \sin x$ so that $du = \cos x\,dx$. We then have

$$\int \frac{1}{\cos x(1-\sin x)^2}\,dx = \int \frac{1}{\cos^2 x(1-u)^2}\,du = \int \frac{1}{(1-u^2)(1-u)^2}\,du$$

Using partial fractional decomposition of the last integrand we obtain

$$\begin{aligned}
\int \frac{1}{(1-u^2)(1-u)^2}\,du &= \int \frac{1}{(1+u)(1-u)^3}\,du \\
&= A\int \frac{1}{1-u}\,du + B\int \frac{1}{(1-u)^2}\,du + C\int \frac{1}{(1-u)^3}\,du + D\int \frac{1}{1+u}\,du \\
&= -A\ln|1-u| + \frac{B}{1-u} + \frac{C}{2(1-u)^2} + D\ln|1+u| + E \\
&= -A\ln|1-\sin x| + \frac{B}{1-\sin x} + \frac{C}{2(1-\sin x)^2} + D\ln|1+\sin x| + E
\end{aligned}$$

To find the coefficients above we need to solve

$$1 = A(1-u)^2(1+u) + B(1-u^2) + C(1+u) + D(1-u)^3.$$

For $u = \pm 1$, we obtain $C = 1/2$ and $D = 1/8$. Since the coefficient of u^3, which is $A - D$ in this case, should be zero, we obtain $A = D$. Finally, we let $u = 0$ to get $B = 1/4$.

3. Evaluate $\displaystyle\int \frac{1}{a\sin x + b\cos x}\,dx$, where $a, b \in \mathbb{R}$, not both zero. (Hint. Use the identity for $\cos(x-\alpha)$ with an appropriate α.)

Multiplying the integrand by a factor of -1 if necessary, we can assume that $a > 0$. If we then let $A = 1/\sqrt{a^2+b^2}$ and denote by α the unique angle in $[0, \pi]$ satisfying $\cos\alpha = bA$, we see that

$$\int \frac{1}{a\sin x + b\cos x}\,dx = \int \frac{A}{\cos(x-\alpha)}\,dx \overset{u=x-\alpha}{=} A\int \sec u\,du = A\ln|\sec(x-\alpha) + \tan(x-\alpha)| + C$$

4. Evaluate $\displaystyle\int \frac{1}{\sqrt{4x-x^2-3}}\,dx$

$$\int \frac{1}{\sqrt{4x-x^2-3}}\,dx = \int \frac{1}{\sqrt{1-(x-2)^2}}\,dx = \arcsin(x-2) + C$$