Spring 2007-08 MATH 112 Homework 4 Solutions

1. Find the limits

$$a) \lim_{n\to\infty} n(a^{1/n}-1),$$

Solution. Change to $y=\frac{1}{n}\to 0$, then use the L'Hopitale's rule

$$\lim_{n\to\infty} n(a^{1/n}-1) = \lim_{y\to 0}\frac{a^y-1}{y} = \lim_{y\to 0}\frac{a^y\ln a}{1} = \ln a.$$

$$b) \lim_{n\to\infty}(n+1)!n^{-n}.$$

Solution 1. Estimate $0 < \dfrac{1\cdot 2\cdot 3...n\cdot(n+1)}{n\cdot n\cdot n...n} \le \dfrac{1\cdot 2\cdot\cdot(n+1)}{n\cdot n\cdot} = \dfrac{2n+2}{n^2}$. Since $\lim_{n\to\infty}\frac{2n+2}{n^2}=0$ then by the sandwich theorem $\lim_{n\to\infty}(n+1)!n^{-n}=0$.

Solution 2. Consider the series $\displaystyle\sum_{n=1}^{\infty}\frac{(n+1)!}{n^n}$ which is convergent (it is easily proved by the ratio test). By the nth test we have

$$\lim_{n\to\infty}(n+1)!n^{-n}=0.$$

2. Find the sum of the series

$$\sum_{n=2}^{\infty}\frac{2n+1}{(n^2-1)(n+2)n}.$$

Solution 1. Since the denominator is a product of the form $(n-1)n(n+1)(n+2)$ try to represent the term $a_n=\frac{2n+1}{(n-1)n(n+1)(n+2)}$ as a difference b_n-b_{n+2}, with $b_n=\frac{A}{(n-1)n}$, where A is a constant to be found.

$$a_n = \frac{2n+1}{(n-1)n(n+1)(n+2)} = \frac{A}{(n-1)n} - \frac{A}{(n+1)(n+2)} =$$

$$= \frac{A(n^2+3n+2)-A(n^2-n)}{(n-1)n(n+1)(n+2)} = \frac{A(4n+2)}{(n-1)n(n+1)(n+2)}.$$

Therefore $A=1/2$ and $a_n=b_n-b_{n+2}$ with $b_n=\frac{1/2}{(n-1)n}$. The series is telescoping and its partial sum S_n is of the form

$$S_n = \sum_{k=2}^{n}(b_k-b_{k+2}) = (b_2-b_4)+(b_3-b_5)+...+(b_n-b_{n+2}) =$$

$$= b_2 + b_3 - b_{n+1} - b_{n+2}.$$

Evidently

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}(b_2+b_3-b_{n+1}-b_{n+2}) = \lim_{n\to\infty}(1/3-\frac{1/2}{n(n+1)}-\frac{1/2}{(n+1)(n+2)}) = \frac{1}{3}.$$

Solution 2. Decompose completely

$$a_n = \frac{2n+1}{(n-1)n(n+1)(n+2)} = \frac{A}{n-1} + \frac{B}{n} + \frac{C}{n+1} + \frac{D}{n+2}.$$

Find the coefficients A, B, C, D by solving the equation

$$An(n+1)(n+2)+B(n^2-1)(n+2)+Cn(n-1)(n+2)+Dn(n^2-1) = 2n+1$$

and get $A = 1/2$, $B = -1/2$, $C = -1/2$, $D = 1/2$. So a_n is represented as $a_n = b_n - b_{n+1}$ with $b_n = \frac{1/2}{(n-1)} - \frac{1/2}{(n+1)} = \frac{1}{n^2-1}$. The series is telescoping and its partial sum S_n is of the form

$$S_n = \sum_{k=2}^{n}(b_k - b_{k+1}) = (b_2 - b_3) + (b_3 - b_4) + ... + (b_n - b_{n+1}) =$$

$$= b_2 - b_{n+1}.$$

Evidently

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}(b_2 - b_{n+1}) = \lim_{n\to\infty}(1/3 - \frac{1}{n^2+2n}) = \frac{1}{3}.$$

3a. Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{2n^2+18n+40},$$

Solution. The series is telescoping because

$$a_n = \frac{1/2}{n+4} - \frac{1/2}{n+5} = b_n - b_{n+1}$$

with $b_n = \frac{1/2}{n+4}$. One can show that $S_n = b_1 - b_{n+1}$. Now find the limit

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}(b_1 - b_{n+1}) = \frac{1}{10} - \lim_{n\to\infty}\frac{1/2}{n+5} = \frac{1}{10}.$$

3b. Determine whether the following series is convergent

$$\sum_{n=3}^{\infty} \frac{1}{\sqrt{n}\ln n + \sin^2 n}.$$

Solution 1. Use the limit comparison test. Compare with the divergent series $\sum_{n=3}^{\infty} \frac{1}{n^{2/3}}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(\sqrt{n}\ln n + \sin^2 n)}{1/n^{2/3}} = \lim_{n\to\infty} \frac{n^{2/3}}{(\sqrt{n}\ln n + \sin^2 n)} = \infty.$$

By the limit comparison test, Part 3, the series diverges.

Solution 2. Use the direct comparison test. We have $\ln x < x$ for $x \geq 1$. So $\ln n = 2\ln\sqrt{n} \geq 2\sqrt{n}$, hence $\sqrt{n}\ln n + \sin^2 n \leq 2n + 1 < 3n$ for $n > 3$. Therefore,

$$\frac{1}{\sqrt{n}\ln n + \sin^2 n} > \frac{1}{3n}.$$

Direct comparison with the divergent harmonic series proves that the series diverges.

4. Determine whether the series are convergent

$$a) \sum_{n=1}^{\infty} n^2 e^{-\sqrt{n}},$$

Solution 1. Compare with the convergent series $\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}/2}}{\sqrt{n}}$. Its convergence is easily checked by the integral test

$$\int_1^{\infty} \frac{e^{-\sqrt{x}/2}}{\sqrt{x}} dx = \lim_{b\to\infty} 4 \int_1^b e^{-\sqrt{x}/2} d(\sqrt{x}/2) = 4 \lim_{b\to\infty} (e^{-1/2} - e^{-\sqrt{b}/2}) = \frac{4}{\sqrt{e}}.$$

Find the limit

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2\sqrt{n}e^{-\sqrt{n}}}{e^{-\sqrt{n}/2}} = \lim_{n\to\infty} n^{5/2} e^{-\sqrt{n}/2} = 0.$$

By the limit comparison test, Part 2, the series converges.

Solution 2. Use the integral test since $f(x) = x^2 e^{-\sqrt{x}}$ is continuous, positive and nondecreasing for $x > 16$. Indeed, $f'(x) = 2xe^{-\sqrt{x}}(1 - \frac{\sqrt{x}}{4}) < 0$ for $x > 16$.

$$\int_{16}^{\infty} x^2 e^{-\sqrt{x}} dx = \int_4^{\infty} 2y^5 e^{-y} dy = \lim_{b\to\infty} \int_4^b 2y^5 e^{-y} dy$$

Find the primitive by integrating by parts several times

$$\int 2y^5 e^{-y} dy = -2e^{-y}(y^5 + 5y^4 + 20y^3 + 60y^2 + 120y + 120).$$

The improper integral converges, because the limit

$$\lim_{b\to\infty} \int_4^b 2y^5 e^{-y} dy = -2 \lim_{b\to\infty} e^{-y}(y^5 + 5y^4 + 20y^3 + 60y^2 + 120y + 120)|_4^b < \infty.$$

Therefore the series converges.

$$b) \sum_{n=1}^{\infty} n \tan \frac{1}{n}.$$

Solution. Find the limit

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n \tan \frac{1}{n} = \lim_{n\to\infty} n \sin \frac{1}{n} \lim_{n\to\infty} 1/\cos \frac{1}{n} = 1 \neq 0$$

By the nth term divergence test the series diverges.
5. Determine whether the following improper integrals are convergent

$$a) \int_0^{\infty} \frac{e^{-x}}{\sqrt{x}} dx.$$

Solution 1. Rewrite as

$$\int_0^{\infty} \frac{e^{-x}}{\sqrt{x}} dx = \int_0^1 \frac{e^{-x}}{\sqrt{x}} dx + \int_1^{\infty} \frac{e^{-x}}{\sqrt{x}} dx$$

The first one is an improper integral of the II type, use DCT for it: the integral converges because $\frac{e^{-x}}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}$ for $x \in [0, 1]$ and the integral $\int_0^1 \frac{1}{\sqrt{x}} dx$ converges. The second is an improper integral of the first type. Compare with the convergent integral $\int_1^{\infty} e^{-x} dx$. Because of the inequality $\frac{e^{-x}}{\sqrt{x}} \leq e^{-x}$ for $x \geq 1$ we can conclude that the integral converges.

Solution 2. To study $\int_0^1 \frac{e^{-x}}{\sqrt{x}} dx$ use LCT, compare with $\int_0^1 \frac{1}{\sqrt{x}} dx$. Find the limit

$$\lim_{x\to 0} \frac{e^{-x}}{\sqrt{x}} / \frac{1}{\sqrt{x}} = \lim_{x\to 0} e^{-x} = 1.$$

By LCT the integral converges. To study the second summand use DCT

$$\frac{e^{-x}}{\sqrt{x}} \leq \frac{e^{-\sqrt{x}}}{\sqrt{x}}.$$

Integral $\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx$ converges, indeed

$$\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx = \int_1^\infty e^{-\sqrt{x}}2d\sqrt{x} = \lim_{b\to\infty} 2\int_1^b e^{-\sqrt{x}}d\sqrt{x} = 2\lim_{b\to\infty}(e^{-1}-e^{-\sqrt{b}}) = 2e^{-1}.$$

Thus by the DCT the integral

$$\int_1^\infty \frac{e^{-x}}{\sqrt{x}}dx$$

converges. Therefore the integral $\int_0^\infty \frac{e^{-x}}{\sqrt{x}}dx$ converges.

$$b) \int_0^\infty \frac{|\sin x|}{x^2}dx.$$

Solution 1. Represent in the form

$$\int_0^\infty \frac{|\sin x|}{x^2}dx = \int_0^1 \frac{|\sin x|}{x^2}dx + \int_1^\infty \frac{|\sin x|}{x^2}dx$$

Concentrate on the first summand, it is an improper integral of type II. Use DCT, from the graph of the function $y = \sin x$ we have that $0 \le \frac{2x}{\pi} \le \sin x$ for $0 \le x \le \pi/2$, hence $0 \le \frac{2}{\pi x} \le \frac{sinx}{x^2}$. Since $\int_0^1 \frac{2}{\pi x}dx$ diverges the integral $\int_0^1 \frac{|\sin x|}{x^2}dx$ also diverges. Therefore the integral $\int_0^\infty \frac{|\sin x|}{x^2}dx$ diverges.

Solution 2. We can use LCT to study the integral $\int_0^1 \frac{|\sin x|}{x^2}dx$. Compare with the divergent integral $\int_0^1 \frac{1}{x}dx$. Indeed

$$\lim_{x\to 0} \frac{\frac{|\sin x|}{x^2}}{\frac{1}{x}} = \lim_{x\to 0} \frac{|\sin x|}{x} = 1.$$

By the limit comparison test the integral diverges.