Solutions of Homework 3

(25 pts) **1.** Let f be a positive increasing continuous function on the interval $[a,b]$ where $a<b$ are positive real numbers. Show that

$$\int_a^b \pi\big(f(x)\big)^2 dx + \int_{f(a)}^{f(b)} 2\pi x f^{-1}(x)dx = b\pi\big(f(b)\big)^2 - a\pi\big(f(a)\big)^2,$$

where f^{-1} denotes the inverse of the invertible function f. (**Hint:** Let R_1 be the region bounded by the curves $x=a$, $x=b$, $y=0$, and $y=f(x)$. Let R_2 be the region bounded by the curves $y=f(a)$, $y=f(b)$, $x=0$, and $y=f(x)$. Consider the volumes of the solids obtained by rotating about the x-axis the regions R_1 and R_2.)

Solution: The volume of the solid obtained by rotating R_1 about the x-axis is equal by the disc method to the integral

$$\boxed{V_1 = \int_a^b \pi\big(f(x)\big)^2 dx.} \quad \text{5pts}$$

The volume of the solid obtained by rotating R_2 about the x-axis is equal by the shell method to the integral

$$\text{6pts} \quad \boxed{V_2 = \int_{f(a)}^{f(b)} 2\pi y f^{-1}(y)dy} = \boxed{\int_{f(a)}^{f(b)} 2\pi x f^{-1}(x)dx.} \quad \text{1pts}$$

It is clear that

$$V_1 + V_2 + \begin{pmatrix}\text{the volume of the cylinder}\\ \text{with the radius } f(a)\\ \text{and the height } a\end{pmatrix} = \begin{pmatrix}\text{the volume of the cylinder}\\ \text{with the radius } f(b)\\ \text{and the height } b\end{pmatrix} \quad \text{7pts}$$

Therefore,

$$\boxed{V_1 + V_2 + \pi\big(f(a)\big)^2 a = \pi\big(f(b)\big)^2 b,} \quad \text{4pts}$$

2pts showing the desired equality.

2. Show that for $\alpha \geq 0$, (25pts)

$$\int_0^\alpha \sqrt{1+4x^2e^{-2x^2}}\,dx \geq \sqrt{\alpha^2 + (1-e^{-\alpha^2})^2}.$$

(**Hint:** Realize the integral in the left hand side of the inequality as the length of the arc of a curve, and realize the right hand side of the inequality as the length of a line segment. Use the fact that the smallest of the lengths of all the paths connecting two points is the length of the line segment connecting these points.)

Solution: The length of the curve $y=g(x)$, where g is a continuously differentiable function, from $x=a$ to $x=b$ is defined to be the integral

$$\boxed{\int_a^b \sqrt{1+\big(g'(x)\big)^2}\,dx.} \quad \text{3pts}$$

Therefore, the length of the curve $y = e^{-x^2}$ from $x = 0$ to $x = \alpha$ is the integral

$$\int_0^\alpha \sqrt{1 + 4x^2 e^{-2x^2}}\,dx,$$

12 pts

which must be greater than or equal to the length of the line segment connecting the points $(0, f(0))$ and $(\alpha, f(\alpha))$ where $f(x) = e^{-x^2}$ 4 pts

Finally, $f(0) = 1$ and $f(\alpha)) = e^{-\alpha^2}$, and the length of the line segment connecting the points $(0, f(0))$ and $(\alpha, f(\alpha))$ is $\sqrt{\alpha^2 + (1 - e^{-\alpha^2})^2}$. 6 pts

(25 pts) **3.** Let $g(x)$ be a positive continuous function on $[0, \infty)$. If for any $a \geq 0$ the area of the region bounded by the curves $x = 0$, $x = a$, $y = 0$, and $y = g(x)$ is $\frac{a^3}{3}$, what is the area of the surface generated by revolving the curve

$$y = \int_0^x g(t)dt, \quad 1 \leq x \leq 2$$

about the x-axis?

Solution: The area of the region bounded by the curves $x = 0$, $x = a$, $y = 0$, and $y = g(x)$ is equal to the integral

$$\boxed{\int_0^a g(x)dx = \int_0^a g(t)dt.}$$ 3 pts

Therefore, for the given curve we have

$$\boxed{y = \int_0^x g(t)dt = \frac{x^3}{3}.}$$ 4 pts

The required surface area is given by the integral

$$S = \int_1^2 2\pi y \sqrt{1 + (\frac{dy}{dx})^2}dx \underbrace{=}_{y=\frac{x^3}{3}} \int_1^2 2\pi \frac{x^3}{3}\sqrt{1 + x^4}dx$$ 5 pts

3 pts

$$\underbrace{=}_{u=1+x^4, du=4x^3dx} \int_2^{17} \frac{\pi}{6}\sqrt{u}du = \frac{\pi}{9}u^{3/2}|_1^{17}.$$ 5 pts 5 pts

(25 pts) **4.** Let $f(x)$ be a positive continuously differentiable function on $[1, \infty]$. For any $a > 1$, let:

$A(a) =$ the area of the region bounded by the curves $x = 1$, $x = a$, $y = 0$, and $y = f(x)$;

$L(a) =$ the length of the curve $y = f(x)$ from $x = 1$ to $x = a$;

$S(a) =$ the area of the surface obtained by rotating about the x-axis the curve $y = f(x)$ from $x = 1$ to $x = a$;

$V(a) =$ the volume of the solid obtained by rotating about the x-axis the region bounded by the curves $x = 1$, $x = a$, $y = 0$, and $y = f(x)$.

Show that

$$\frac{d^2V}{da^2}\frac{dL}{da} = \frac{d^2A}{da^2}\frac{dS}{da}.$$

Solution: We have that

2pts

$$A(a) = \int_1^a f(x)dx, \qquad L(a) = \int_1^a \sqrt{1+(f'(x))^2}\,dx,$$ 2pts

2pts $$S(a) = \int_1^a 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx, \qquad V(a) = \int_1^a \pi(f(x))^2 dx.$$ 2pts

Now, the fundamental theorem of calculus implies that

4pts $$\frac{d^2A}{da^2} = \frac{d}{da}\Big(\frac{d}{da}\int_1^a f(x)dx\Big) = \frac{d}{da}f(a) = f'(a),$$

6pts $$\frac{d^2V}{da^2} = \frac{d}{da}\Big(\frac{d}{da}\int_1^a \pi(f(x))^2dx\Big) = \frac{d}{da}\Big(\pi(f(a))^2\Big) = 2\pi f(a)f'(a),$$

3pts $$\frac{dS}{da} = \frac{d}{da}\Big(\int_1^a 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx\Big) = 2\pi f(a)\sqrt{1+(f'(a))^2},$$

2pts $$\frac{dL}{da} = \frac{d}{da}\Big(\int_1^a \sqrt{1+(f'(x))^2}\,dx\Big) = \sqrt{1+(f'(a))^2}.$$

The result follows.

2pts