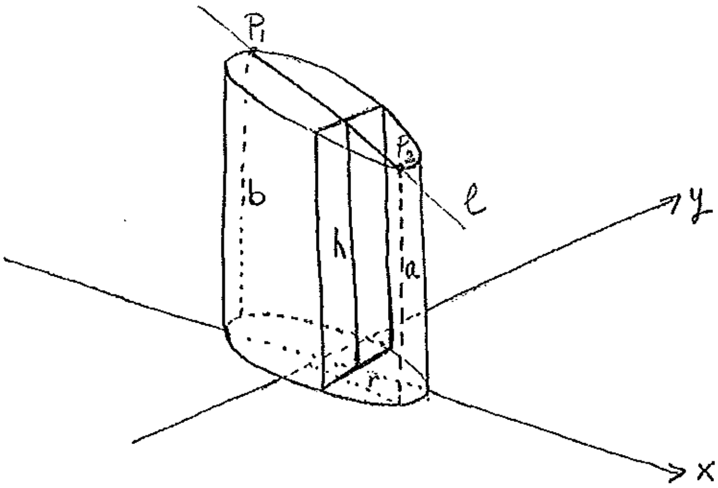
Math.112 ,HW1
Due on Feb. 16,2009 up to 12:40 p.m.

(25 pts.) 1. The top of a circular cylinder of radius r is a plane inclined at an angle to the horizontal as shown in the figure. If the lowest and highest points on the top are at heights a and b, respectively, above the base, find the volume of the cylinder.
Note: There is an easy geometric way to get the answer, but you should also try to do it by slicing. You can use rectangular slices.



Let the x-axis be along the diameter shown in the figure, with the origin at the center of the base. The cross-section perpendicular to the x-axis at x is a rectangle having base $2\sqrt{r^2-x^2}$ and height $h = \frac{a+b}{2} + \frac{a-b}{2r}x$, why?
(The height h is determined by the line ℓ which passes through the points $P_1(-r, b)$ and $P_2(r, a)$, i.e., $\ell: y = \frac{a+b}{2} + \frac{a-b}{2r}x$.)

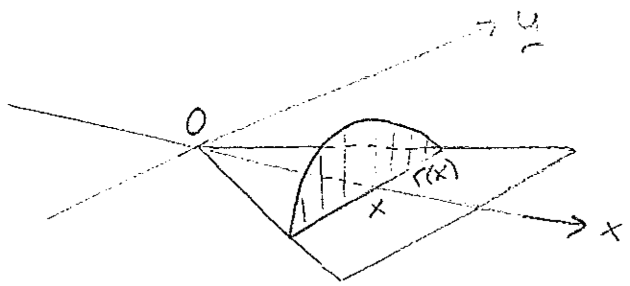
Thus the volume of the solid of the described cylinder is

$$V = \int_{-r}^{r} \left(2\sqrt{r^2 - x^2}\right)\left(\frac{a+b}{2} + \frac{a-b}{2r}x\right)dx$$

$$= (a+b)\underbrace{\int_{-r}^{r}\sqrt{r^2-x^2}\,dx}_{=\frac{\pi r^2}{2} \text{ which is the area of the related semicircle}} + \frac{a-b}{2r}\underbrace{\int_{-r}^{r} x\sqrt{r^2-x^2}\,dx}_{=0 \text{ since the integrand fcn. is odd}}$$

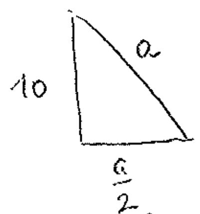
$$= \frac{\pi r^2 (a+b)}{2} \text{ cubic units.}$$

(25 pts.)2. Let the base of the solid be the equilateral triangle with altitude 10 cm. Suppose that the cross-sections perpendicular both to the base and to a given altitude of the triangle are semicircles. Find the volume of the solid.



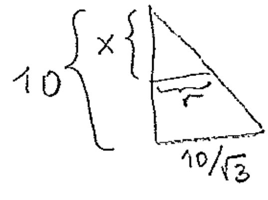
Let the radius of the semicircular cross-section x units from the vertex O shown in the figure be r(x) units long. Then

Remember:



$(10)^2 + \left(\frac{a}{2}\right)^2 = a^2$

$\Rightarrow a = \frac{20}{\sqrt{3}}$



$\frac{x}{10} = \frac{r(x)}{10/\sqrt{3}}$

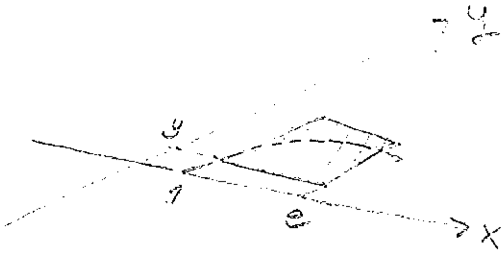
$\Rightarrow r(x) = \frac{x}{\sqrt{3}}$.

Thus the cross-sectional area is given by

$A(x) = \frac{\pi}{2}(r(x))^2 = \frac{\pi x^2}{6}$, so

$$V = \int_0^{10} \frac{\pi x^2}{6}\,dx = \left.\frac{\pi x^3}{18}\right|_{x=0}^{10} = \frac{500\pi}{9} \text{ cu. units}$$

(25 pts.)3. The base of a solid S is in the region between the x-axis and the curve $y = \ln x$ from $x = 1$ to $x = e$. Find the volume of S if every cross-section of S by a plane perpendicular to the $y-$axis is a square.



$$A(y) = (e - e^y)^2, \quad y \in [0,1]$$

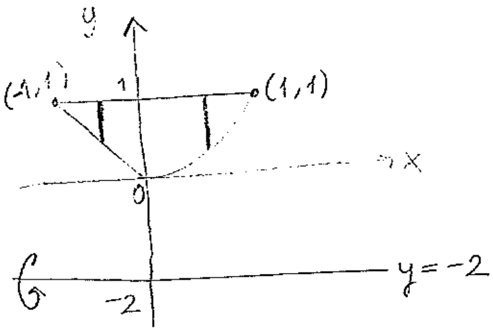
$$V = \int_0^1 (e - e^y)^2 dy$$

$$= \int_0^1 (e^2 - 2ee^y + e^{2y}) dy$$

$$= e^2 y - 2e^{y+1} + \frac{e^{2y}}{2} \Big|_{y=0}^{1}$$

$$= \frac{4e - e^2 - 1}{2} \text{ cu. units.}$$

(25 pts.)4. Use the washer (disk) method to find the volume of the solid of revolution obtained by rotating the region bounded by the curves $y = x^3$, $y = -x$, and $y = 1$ around the line $y = -2$.



$$V = \pi \int_{-1}^{0} \left[((1) - (-2))^2 - (-x - (-2))^2 \right] dx$$

$$+ \pi \int_0^1 \left[((1) - (-2))^2 - (x^3 - (-2))^2 \right] dx$$

$$= \pi \int_{-1}^{0} (5 - x^2 + 4x) dx + \pi \int_0^1 (5 - x^6 - 4x^3) dx$$

$$= \frac{137\pi}{21} \text{ cu. units.}$$