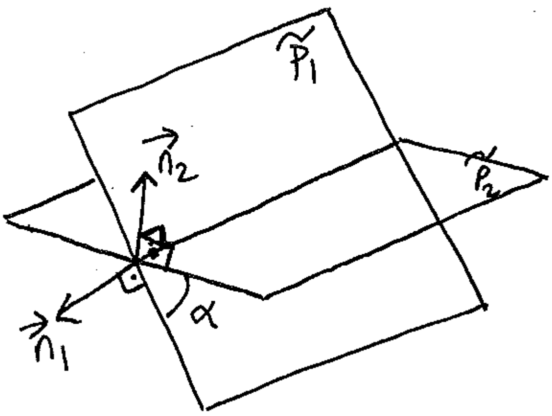
Solutions of Homework 11

1. Find the acute angle between the planes

$$10x - 11y + 2z + 15 = 0 \qquad \text{and} \qquad 5x - 14y + 2z - 8 = 0.$$

Solution:



The acute angle α between the planes $\tilde{P}_1: 10x - 11y + 2z + 15 = 0$ and $\tilde{P}_2: 5x - 14y + 2z - 8 = 0$ is equal to the acute angle between their normal vectors $\vec{n}_1$ and $\vec{n}_2$.

A normal vector to the plane $\tilde{P}_1$ is $\vec{n}_1 = \langle 10, -11, 2 \rangle$, and a normal vector of the plane $\tilde{P}_2$ is $\vec{n}_2 = \langle 5, -14, 2 \rangle$.

$$\vec{n}_1 \cdot \vec{n}_2 = 10(5) + (-11)(-14) + 2(2) = 208$$

$$|\vec{n}_1| = \sqrt{10^2 + (-11)^2 + 2^2} = \sqrt{225} = 15$$

$$|\vec{n}_2| = \sqrt{5^2 + (-14)^2 + 2^2} = \sqrt{225} = 15$$

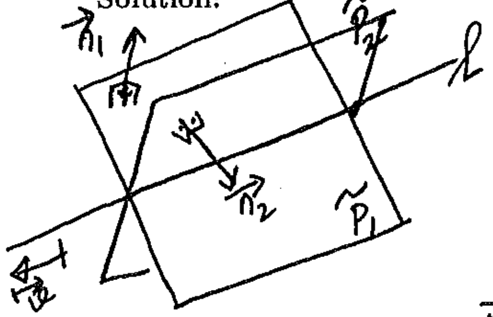
As $\vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1| |\vec{n}_2| \cos\theta$ where θ is the angle between $\vec{n}_1$ and $\vec{n}_2$, $\cos\theta = \dfrac{208}{225} > 0$ so that θ is an acute angle. Hence, the acute angle between the given planes is

$$\alpha = \theta = \arccos\left(\frac{208}{225}\right).$$

2. Find an equation of the line of intersection of the planes

$$4x - y - z = -2 \qquad \text{and} \qquad x + y - z = 5.$$

Solution:



The planes $\tilde{P}_1: 4x-y-z=-2$ and $\tilde{P}_2: x+y-z=5$ intersect along a line ℓ. A direction vector $\vec{v}$ of ℓ must be orthogonal to normal vectors $\vec{n}_1$ and $\vec{n}_2$ of the planes $\tilde{P}_1$ and $\tilde{P}_2$. So, a direction vector $\vec{v}$ of ℓ may be found as $\vec{v} = \vec{n}_1 \times \vec{n}_2$.

Here, $\vec{n}_1 = \langle 4,-1,-1\rangle$ and $\vec{n}_2 = \langle 1,1,-1\rangle$ so that

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 4 & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ 1 & -1 \end{vmatrix} i - \begin{vmatrix} 4 & -1 \\ 1 & -1 \end{vmatrix} j + \begin{vmatrix} 4 & -1 \\ 1 & 1 \end{vmatrix} k$$

$$= 2i + 3j + 5k = \langle 2,3,5\rangle$$

We obtained a direction vector $\vec{v}$ of ℓ. To write an equation of ℓ we need to find a point on ℓ. So we need to find a point that lies on both planes $\tilde{P}_1$ and $\tilde{P}_2$.

$$\tilde{P}_1 \cap \tilde{P}_2 : \begin{Bmatrix} 4x-y-z=-2 \\ x+y-z=5 \end{Bmatrix} \Rightarrow \begin{Bmatrix} z = 4x-y+2 \\ z = x+y-5 \end{Bmatrix} \Rightarrow 3x-2y+7=0$$

For example, $x=-1$ and $y=2$ satisfies the equation $3x-2y+7=0$, and if we substitute $x=-1$ and $y=2$ in either of the equations for the planes $\tilde{P}_1$ and $\tilde{P}_2$ we find that $z=-4$. Thus, the point $(-1,2,-4)$ is on the line ℓ.

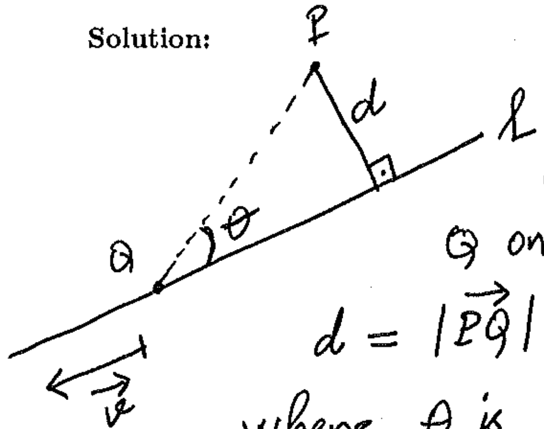
Hence, ℓ is a line through the point $(-1,2,-4)$ and parallel to the vector $\vec{v} = \langle 2,3,5\rangle$, so that an equation of ℓ is

$$x = -1+2t, \quad y = 2+3t, \quad z = -4+5t \qquad (t \in \mathbb{R})$$

3. Find the distance from the point $(1,2,3)$ to the line

$$x-2=\frac{2-y}{3}=\frac{z}{5}$$

Solution:

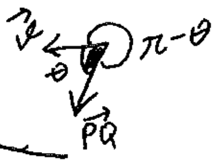


To find the distance d from the point P to the line ℓ with a direction vector $\vec{v}$, we take a point Q on the line ℓ, and we note that

$$d = |\vec{PQ}|\sin\theta = \frac{|\vec{PQ}|\,|\vec{v}|\sin\theta}{|\vec{v}|} = \frac{|\vec{PQ}\times\vec{v}|}{|\vec{v}|}$$

where θ is an between the vectors $\vec{PQ}$ and $\vec{v}$.

Notice here that, as $\sin(\theta)=\sin(\pi-\theta)$, it does not matter whether θ is the angle between $\vec{PQ}$ and $\vec{v}$ or $\vec{v}$ and $\vec{PQ}$



In our question, $\ell: x-2=\frac{2-y}{3}=\frac{z}{5}$, and so a direction vector of ℓ is $\vec{v}=\langle 1,-3,5\rangle$ and $P=(1,2,3)$. We take a point Q on ℓ, for instance $Q(2,2,0)$. Thus,

$\vec{PQ}=\langle 1,0,-3\rangle$ and so

$$\vec{PQ}\times\vec{v}=\begin{vmatrix} i & j & k \\ 1 & 0 & -3 \\ 1 & -3 & 5\end{vmatrix} = \begin{vmatrix} 0 & -3 \\ -3 & 5\end{vmatrix} i - \begin{vmatrix} 1 & -3 \\ 1 & 5\end{vmatrix} j + \begin{vmatrix} 1 & 0 \\ 1 & -3\end{vmatrix} k$$

$$= -9i-8j-3k = \langle -9,-8,-3\rangle$$

$$|\vec{PQ}\times\vec{v}| = \sqrt{(-9)^2+(-8)^2+(-3)^2} = \sqrt{154}$$

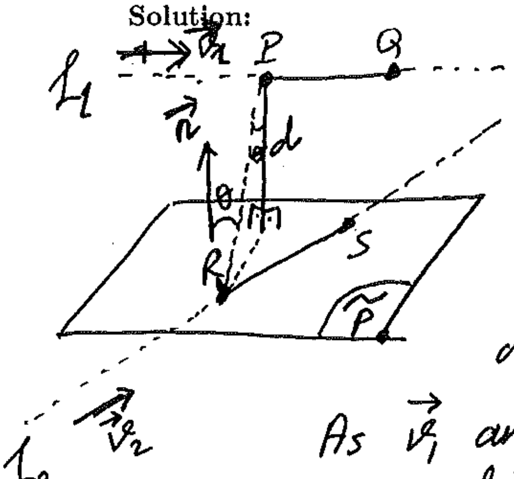
$$|\vec{v}| = \sqrt{1^2+(-3)^2+5^2} = \sqrt{35}$$

So, the required distance is

$$d=\frac{\sqrt{154}}{\sqrt{35}}=\sqrt{\frac{22}{7}}$$

4. Find the distance between the line through $P(6,-8,0)$ and $Q(2,-1,-1)$ and the line through $R(0,2,-1)$ and $S(2,1,2)$.

Solution:



Let L_1 be the line through P and Q, and let L_2 be the line through R and S. $\vec{v}_1 = \vec{PQ} = \langle -4, 7, -1\rangle$ is a direction vector of L_1, and $\vec{v}_2 = \vec{RS} = \langle 2, -1, 3\rangle$ is a direction vector of L_2.

As $\vec{v}_1$ and $\vec{v}_2$ are not parallel (because they are not proportional), the lines L_1 and L_2 are <u>not</u> parallel.

Equations of the lines are $\begin{cases} L_1: x = 6-4t,\ y = -8+7t,\ z = -t \\ L_2: x = 2s,\ y = 2-s,\ z = -1+3s \end{cases}$

$\underline{L_1 \cap L_2 = \phi}$: If the lines L_1 and L_2 intersect at a point, the coordinates (x,y,z) of this point must be

$x = 6-4t_0 = 2s_0$, $y = -8+7t_0 = 2-s_0$, $z = -t_0 = -1+3s_0$

for some real numbers t_0 and s_0. These x-, y-, z-coordinates gives

$\begin{cases} 2s_0 + 4t_0 = 6 \\ s_0 + 7t_0 = 10 \\ 3s_0 + t_0 = 1 \end{cases}$. $(1^{st}$ eqn.$) + (2^{nd}$ eqn.$) - (3^{rd}$ eqn$)$ gives that $10t_0 = 15$ so that $t_0 = 3/2$.

Substituting $t_0 = 3/2$ in these three equations we get that

$s_0 = 0$, $s_0 = -1/2$, $s_0 = -1/6$.

Thus, the lines L_1 and L_2 do <u>not</u> intersect.

Hence, the lines L_1 and L_2 are skew lines.

To find the distance between the skew lines l_1 and l_2, we may first find a plane $\tilde{P}$ containing l_2 and parallel to l_1, and then we may find the distance from a point on l_1 to the plane $\tilde{P}$. This distance is equal to the distance between l_1 and l_2.

A normal vector $\vec{n}$ of $\tilde{P}$ is orthogonal to direction vectors $\vec{v}_1$ and $\vec{v}_2$ of l_1 and l_2. Hence,

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} i & j & k \\ -4 & 7 & -1 \\ 2 & -1 & 3 \end{vmatrix} = \begin{vmatrix} 7 & -1 \\ -1 & 3 \end{vmatrix} i - \begin{vmatrix} -4 & -1 \\ 2 & 3 \end{vmatrix} j + \begin{vmatrix} -4 & 7 \\ 2 & -1 \end{vmatrix} k$$

$$= 20i + 10j - 10k = \langle 20, 10, -10 \rangle$$

is a normal vector of the plane $\tilde{P}$.

The distance d between the lines l_1 and l_2 is equal to the distance from a point (say P) on l_1 to the plane $\tilde{P}$. To find it we take a point (say R) on the plane $\tilde{P}$, and ...... (see the figure).

$$d = \left| |\vec{PR}| \cos\theta \right| = \left| \frac{|\vec{PR}|\,|\vec{n}| \cos\theta}{|\vec{n}|} \right| , \quad (\theta = \text{the angle between the vectors } \vec{PR} \text{ and } \vec{n})$$

$$= \left| \frac{\vec{PR} \cdot \vec{n}}{\vec{n}} \right| .$$

$\vec{PR} = \langle -6, 10, -1 \rangle$, $\quad \vec{n} = \langle 20, 10, -10 \rangle$

$$\vec{PR} \cdot \vec{n} = (-6)20 + 10(10) + (-1)(-10) = -10$$

$$|\vec{n}| = \sqrt{(20)^2 + (10)^2 + (-10)^2} = \sqrt{600} = 10\sqrt{6}$$

So, the required distance is

$$d = \left| \frac{-10}{10\sqrt{6}} \right| = \frac{1}{\sqrt{6}}$$