4)a)Find the **first four nonzero terms** of the Maclaurin series of $F(x)=\int_0^x \sin(t^2)\cos t\,dt$.

$$\sin(t^2)=\sum_{n=0}^{\infty}(-1)^n\frac{t^{4n+2}}{(2n+1)!}=t^2-\frac{t^6}{3!}+\frac{t^{10}}{5!}-\dots\quad ,\forall t$$

$$\cos t=\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{(2n)!}=1-\frac{t^2}{2!}+\frac{t^4}{4!}-\frac{t^6}{6!}+\dots\quad ,\forall t$$

$$\sin(t^2)\cos t=t^2-\frac{t^4}{2!}+\left(\frac{1}{4!}-\frac{1}{3!}\right)t^6+\left(\frac{1}{2!\,3!}-\frac{1}{6!}\right)t^8+\dots$$

$$F(x)=\int_0^x\left(t^2-\frac{t^4}{2}-\frac{t^6}{8}+\frac{59t^8}{720}+\dots\right)dt$$

$$=\frac{x^3}{3}-\frac{x^5}{10}-\frac{x^7}{56}+\frac{59x^9}{6480}+\dots$$

b) Find the exact sum of $\frac{1}{e^2}-\frac{1}{2e^4}+\frac{1}{3e^6}-\dots\dots\dots+(-1)^{n-1}\frac{1}{ne^{2n}}+\dots\dots\dots$

Remember: $$\ln(1+x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}\quad ,|x|<1$$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}\quad ,|x|<1$$

$$=x-\frac{x^2}{2}+\frac{x^3}{3}-\dots+(-1)^{n-1}\frac{x^n}{n}+\dots$$

$$\frac{1}{e^2}-\frac{1}{2e^4}+\frac{1}{3e^6}-\dots+(-1)^{n-1}\frac{1}{ne^{2n}}+\dots=\ln(1+x)\Big|_{x=\frac{1}{e^2}}$$

$$=\ln\left(1+\frac{1}{e^2}\right)$$

$$=\ln(e^2+1)-2.$$