1. (a) Let f be a continuously differentiable function satisfying $f(0) = f(-1) = 1$. Evaluate the definite integral $\int_{-1}^{0} e^{-x}(f'(x) - f(x))\,dx$.

$$\int_{-1}^{0} e^{-x} f'(x)\,dx \overset{\text{by parts}}{=} e^{-x} f(x)\Big|_{-1}^{0} + \int_{-1}^{0} f(x)e^{-x}\,dx = (1-e) + \int_{-1}^{0} f(x)e^{-x}\,dx$$

$u = e^{-x} \Rightarrow du = -e^{-x}dx$

$dv = f'(x)dx \Rightarrow v = f(x)$

$$\Rightarrow \underbrace{\int_{-1}^{0} e^{-x} f'(x)\,dx - \int_{-1}^{0} f(x)e^{-x}\,dx}_{\displaystyle \| \atop \displaystyle \int_{-1}^{0} e^{-x}(f'(x) - f(x))\,dx} = 1 - e$$

(b) Evaluate the indefinite integral $\int \frac{x^5}{x^4+1}\,dx$.

$$\int \frac{x^5}{x^4+1}\,dx \underset{u=x^2,\ du=2x\,dx}{=} \int \frac{x^4\,x\,dx}{x^4+1} = \int \frac{u^2\left(\frac{1}{2}du\right)}{u^2+1} = \frac{1}{2}\int \frac{u^2}{u^2+1}\,du$$

$$= \frac{1}{2}\int \frac{(u^2+1)-1}{u^2+1}\,du = \frac{1}{2}\int \left(1 - \frac{1}{u^2+1}\right)du$$

$$= \frac{1}{2}(u - \arctan u) + C$$

$$\underset{u=x^2}{=} \frac{1}{2}x^2 - \frac{1}{2}\arctan(x^2) + C$$

2. (a) Let f be a continuous function such that $f(0) = 7$. Is the series $\sum_{n=1}^{\infty} f\left(\frac{1}{n}\right)$ convergent? Explain your answer.

$$\lim_{n\to\infty} f\left(\frac{1}{n}\right) \underset{\substack{\uparrow \\ f \text{ is continuous}}}{=} f\left(\lim_{n\to\infty} \frac{1}{n}\right) = f(0) = 7 \neq 0$$

The "n^{th} term test" implies that the series $\sum f\left(\frac{1}{n}\right)$ is divergent.

(b) Suppose that $\lim_{n\to\infty} n^{3/2} a_n = 1$ where $a_n \geq 0$ for all n. Does the series $\sum_{n=1}^{\infty} a_n$ converge? Explain your answer.

$$1 = \lim_{n\to\infty} n^{3/2} a_n = \lim_{n\to\infty} \frac{a_n}{\frac{1}{n^{3/2}}}$$

. As $1 \in (0, \infty)$ and as $a_n \geq 0$, the "limit comparison test" implies that $\sum a_n$ and $\sum \frac{1}{n^{3/2}}$ both converge or both diverge.

Since $\sum \frac{1}{n^{3/2}}$ converges (because $p = 3/2 > 1$), the series $\sum a_n$ converges too.

3. Prove that $\int_0^1 \frac{\ln(1+x)}{x}dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ and determine how many terms should be used to estimate this sum with an error less that 10^{-2}.

For $|x|<1$, $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ so that $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$. Integrating term by term,
$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$ which has no constant term.

Thus, $\frac{\ln(1+x)}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}$ for $|x|<1$.

term by term integration

$$\int_0^1 \frac{\ln(1+x)}{x}dx = \int_0^1 \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}\right)dx = \sum_{n=0}^{\infty}\left(\int_0^1 (-1)^n \frac{x^n}{n+1}dx\right)$$
$$= \sum_{n=0}^{\infty}\left((-1)^n \frac{x^{n+1}}{(n+1)^2}\Bigg|_{x=0}^{x=1}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n+1)^2}$$
$$= \sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n^2} = \sum_{n=1}^{\infty} (-1)^{n-1}(-1)^2\frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

Let $u_n = \frac{1}{n^2} > 0$. Then $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \sum_{n=1}^{\infty} (-1)^{n+1} u_n$ is an "alternating series"

Note that $u_n > 0$, u_n is decreasing, and $u_n \to 0$ so that we may (and will do) use the "alternating series estimation thm."

The error in truncating $\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - \dots$ after $(-1)^{n+1} u_n$ is no greater than $u_{n+1} = \frac{1}{(n+1)^2}$. Thus,

$\left(|\text{error}| < \frac{1}{(n+1)^2} \le 10^{-2}\right.$ implying that $n \geq 9$.

As a result, (at least) 9 terms must be used to estimate the sum with an error less than 10^{-2}.

4. (a) If $f(x) = \int_0^x t^2 e^{-t^2}\,dt$, find $f^{(17)}(0)$ and $f^{(18)}(0)$.

As $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ (for all x),

$$f(x) = \int_0^x t^2 e^{-t^2} dt = \int_0^x t^2 \left(\sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} \right) dt = \int_0^x \left(\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+2}}{n!} \right) dt$$

$$\underset{\text{term by term integration of power series}}{=} \sum_{n=0}^{\infty} \left(\int_0^x \frac{(-1)^n t^{2n+2}}{n!} dt \right) = \sum_{n=0}^{\infty} \left(\frac{(-1)^n t^{2n+3}}{n!\,(2n+3)} \Bigg|_{t=0}^{t=x} \right) = \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{n!\,(2n+3)}}_{S}$$

For any $k \in \mathbb{N}$, $\frac{f^{(k)}(0)}{k!}$ is the coefficient of x^k in the Maclaurin series generated by f. Hence, $f^{(18)}(0) = 0$ because the coefficient of x^{18} in the series S is 0, and (as $2n+3 = 17 \Rightarrow n = 7$) $f^{(17)}(0)/17! = \frac{(-1)^7}{7!\,17}$ so that $f^{(17)}(0) = -\frac{16!}{7!}$

(b) Evaluate the sum $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^n}{3^n n}$.

Recall that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$. So now, for $|x| < 1$,

$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$. Term by term integration of power series gives that

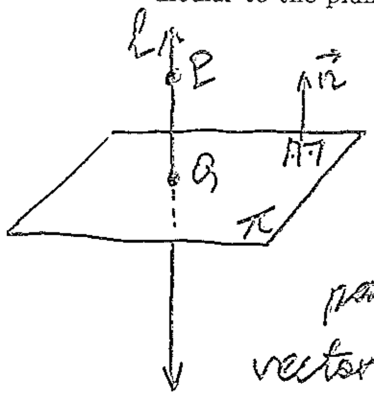
$\ln(1+x) + C = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$. Putting $x = 0$ we see that $C = 0$. So

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

let $x = \frac{2}{3}$ ($!\ |\frac{2}{3}| < 1$) to get,

$$\ln(5/3) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(2/3)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^n}{3^n n}$$

5. (a) Write an equation of the line ℓ passing through the point $P(1, 0, -2)$ and perpendicular to the plane $\pi : x + y + z = 120$.



A normal vector of the plane π is $\vec{n} = \langle 1, 1, 1 \rangle$. As $\ell \perp \pi$, $\vec{n}$ is a direction vector of the line ℓ. So, an equation of the line ℓ passing through the point P and parallel to the vector $\vec{n}$ is

$$x = 1 + t, \quad y = t, \quad z = -2 + t \quad (t \in \mathbb{R})$$

(b) Find the point Q of intersection of the line ℓ and the plane π.

$$\pi \cap \ell : \left\{ \begin{array}{l} x + y + z = 120 \\ x = 1 + t,\ y = t,\ z = -2 + t \end{array} \right\} \Rightarrow \underbrace{(1+t) + t + (-2+t)}_{3t - 1} = 120$$

$$3t - 1 = 120$$

$$\Rightarrow t = \frac{121}{3}$$

So, $x = 1 + t = \frac{124}{3}$, $y = \frac{121}{3}$, $z = \frac{115}{3}$.

Hence, $Q\left(\frac{124}{3}, \frac{121}{3}, \frac{115}{3}\right)$

(c) Find the distance between the point P and intersection point Q.

$$d = |PQ| = |\overrightarrow{PQ}| = \left| \left\langle \frac{124}{3} - 1, \frac{121}{3}, \frac{115}{3} + 2 \right\rangle \right|$$

$$= \left| \left\langle \frac{121}{3}, \frac{121}{3}, \frac{121}{3} \right\rangle \right| = \sqrt{\left(\frac{121}{3}\right)^2 + \left(\frac{121}{3}\right)^2 + \left(\frac{121}{3}\right)^2}$$

$$= \frac{121}{3}\sqrt{3} = \frac{121}{\sqrt{3}}$$