Math. 112 (2009–10, Spring)
Final Key

Question 1 (10+10=20 points)

Let Π be the plane through the points $A(1, 0, -5)$, $B(2, 3, 9)$, and $C(-2, 1, -17)$.

a. Find an equation for the plane Π.

$\overrightarrow{AB} = \langle 1, 3, 14 \rangle \qquad \overrightarrow{AC} = \langle -3, 1, -12 \rangle$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 14 \\ -3 & 1 & -12 \end{vmatrix} = -50\vec{i} - 30\vec{j} + 10\vec{k}$$

An equation for the plane Π

$$-50(x-1) - 30(y-0) + 10(z+5) = 0$$

in other words

$$5x + 3y - z = 10$$

b. Find the angle θ between the plane Π and the line passing through A and $D(1, 1, 1)$.

$\vec{n} = \langle 5, 3, -1 \rangle$ is normal to Π and $\overrightarrow{AD} = \langle 0, 1, 6 \rangle$

$$\theta = \arccos\left(\frac{\vec{n} \cdot \overrightarrow{AD}}{|\vec{n}|\, |\overrightarrow{AD}|} \right) = \arccos\left(\frac{-3}{\sqrt{37}\sqrt{35}} \right)$$

Question 2 (10+10=20 points)

a. Find the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(5x+1)^{4n+1}}{n3^n}$.

$$\lim_{n\to\infty} \frac{\left|\frac{(5x+1)^{4n+5}}{(n+1)3^{n+1}}\right|}{\left|\frac{(5x+1)^{4n+1}}{n\,3^n}\right|} = \frac{|5x+1|^4}{3} < 1$$

$$\Updownarrow$$

$$\frac{-\sqrt[4]{3}-1}{5} < x < \frac{\sqrt[4]{3}-1}{5}$$

$$x = \frac{-\sqrt[4]{3}-1}{5} \Rightarrow \sum_{n=1}^{\infty} \frac{(5x+1)^{4n+1}}{n3^n} = 3\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \quad \text{converges}$$

$$x = \frac{\sqrt[4]{3}-1}{5} \Rightarrow \sum_{n=1}^{\infty} \frac{(5x+1)^{4n+1}}{n3^n} = 3\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges}$$

$$\text{Interval of convergence} = \left[\frac{-\sqrt[4]{3}-1}{5}, \frac{\sqrt[4]{3}-1}{5}\right)$$

b. Compute $\sum_{n=1}^{\infty} \frac{(-1)^n n(n+1)}{5^n}$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{if } -1<x<1$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \sum_{n=1}^{\infty} n x^{n-1} \quad \text{if } -1<x<1$$

$$\frac{x^2}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n+1} \quad \text{if } -1<x<1$$

$$\frac{2x}{(1-x)^3} = \frac{d}{dx}\left(\frac{x^2}{(1-x)^2}\right) = \sum_{n=1}^{\infty} n(n+1)x^n \quad \text{if } -1<x<1$$

Put $x = \frac{-1}{5}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n n(n+1)}{5^n} = \frac{2\left(\frac{-1}{5}\right)}{\left(1+\frac{1}{5}\right)^3}$$

$$= \frac{-1}{25\cdot 36\cdot 3}$$

Question 3 (8+8+8=24 points)

Evaluate the following integrals

a. $\int \frac{dx}{x^3 + x^2 + x}$.

First note $x^3+x^2+x = x(x^2+x+1)$

Second note that

$$\frac{1}{x(x^2+x+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1} \iff 1 = A(x^2+x+1) + (Bx+C)x \iff$$

$$\iff 1 = (A+B)x^2 + (A+C)x + A \iff A=1,\ B=-1,\ C=-1$$

Now $$\int \frac{dx}{x^3+x^2+x} = \int \frac{dx}{x(x^2+x+1)} = \int \left(\frac{1}{x} + \frac{-x-1}{x^2+x+1}\right) dx =$$

$$= \ln|x| - \int \frac{x+\frac{1}{2}}{x^2+x+1}\,dx - \int \frac{\frac{1}{2}}{x^2+x+1}\,dx = \ln|x| - \frac{1}{2}\ln|x^2+x+1|$$

$$-\frac{1}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right) + C$$

b. $\int \cos^7(x)\,dx$.

$$\int \cos^7(x)\,dx = \int \cos^6(x)\cos(x)\,dx = \int (1-\sin^2(x))^3 \cos(x)\,dx =$$

$$= \int (1-u^2)^3\,du = \int (1-3u^2+3u^4-u^6)\,du =$$

$u = \sin(x)$
$du = \cos(x)\,dx$

$$= u - u^3 + \frac{3}{5}u^5 - \frac{1}{7}u^7 + C$$

$$= \sin(x) - \sin^3(x) + \frac{3}{5}\sin^5(x) - \frac{1}{7}\sin^7(x) + C$$

c. $\int \cos(3\ln x)\,dx$. Say $I = \int \cos(3\ln x)\,dx$

$u = \cos(3\ln x) \qquad dv = dx$

$du = -\sin(3\ln x)\frac{3}{x} \qquad v = x$

$I = x\cos(3\ln x) + 3\int \sin(3\ln x)\,dx$

$u = \sin(3\ln x) \qquad dv = dx$

$du = \cos(3\ln x)\frac{3}{x} \qquad v = x$

$I = x\cos(3\ln x) + 3x\sin(3\ln x) - 9I \Rightarrow I = \frac{x(\cos(3\ln x) + 3\sin(3\ln x))}{10} + C$

Question 4 (20 points)

Let $f(x) = \int_0^x e^{-u^4}du$. Find the least positive integer n for which you can prove that the Taylor polynomial $P_n(x)$ of order n generated by $f(x)$, at $x = 0$, approximates the function $f(x)$ on the closed interval $[0, \frac{1}{2}]$ with an error less than 10^{-8}.

For any u we have $e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$

$$f(x) = \int_0^x e^{-u^4}du = \int_0^x \left(\sum_{n=0}^{\infty} \frac{(-u^4)^n}{n!}\right)du = \sum_{n=0}^{\infty} \frac{(-1)^n u^{4n+1}}{n!(4n+1)}\Bigg|_{u=0}^{x} =$$

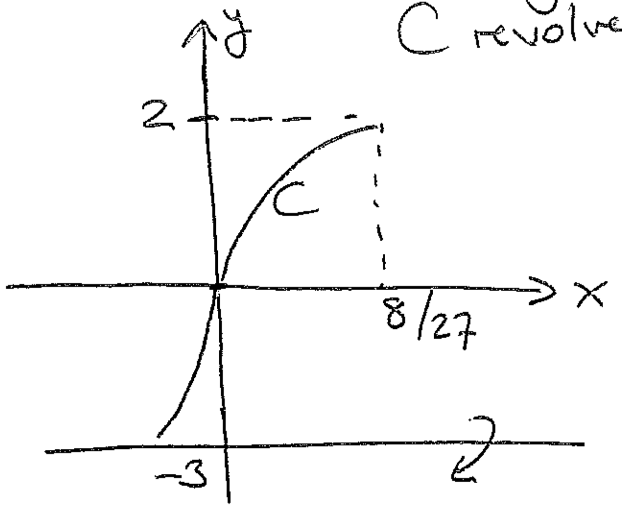
$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{n!(4n+1)}$$

By Alternating Series Remainder Estimation Thm $|R_n(x)| \le \left|\frac{(-1)^{n+1} x^{4n+5}}{(n+1)!(4n+5)}\right| \le \frac{1}{(n+1)!(4n+5)2^{4n+5}}$

Hence $|R_4(x)| \le \frac{1}{5!\,(21)\,2^{21}} < 10^{-8}$

Then $f(x) \approx P(x) = x - \frac{x^5}{5} + \frac{x^9}{18} - \frac{x^{13}}{78} + \frac{x^{17}}{408}$ with error $< 10^{-8}$

5. a) $C: 27x - y^3 = 0$ between $y=0$ and $y=2$.
C revolved about $y=-3$. Surface area = ?



$C: y = \sqrt[3]{27x} = 3x^{1/3}$

$$S = \int_a^b 2\pi \rho \, ds$$

$$\rho = y - (-3) = y + 3 = 3x^{1/3} + 3$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \left(\cancel{3} \cdot \frac{1}{\cancel{3}} x^{-2/3}\right)^2}\, dx$$

$$ds = \sqrt{1 + x^{-4/3}}\, dx$$

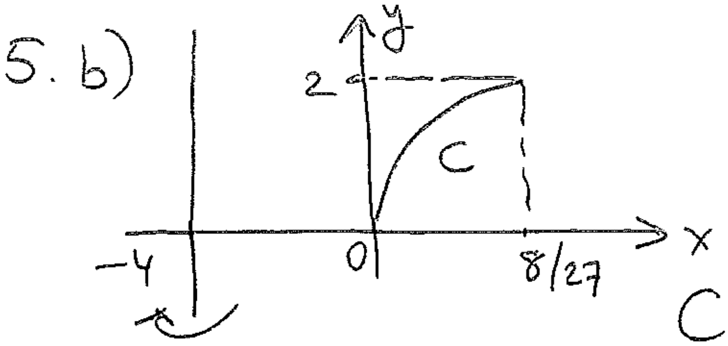
$$S = 2\pi \int_0^{8/27} \left(3x^{1/3} + 3\right) \sqrt{1 + x^{-4/3}}\, dx$$

or with $C: x = \dfrac{y^3}{27}$, $ds = \sqrt{1 + \left(\dfrac{dx}{dy}\right)^2}\, dy$

$$ds = \sqrt{1 + \left(\frac{1}{27} 3y^2\right)^2}\, dy = \sqrt{1 + \frac{y^4}{81}}\, dy,$$

$$S = 2\pi \int_0^2 (y+3) \sqrt{1 + \frac{y^4}{81}}\, dy$$

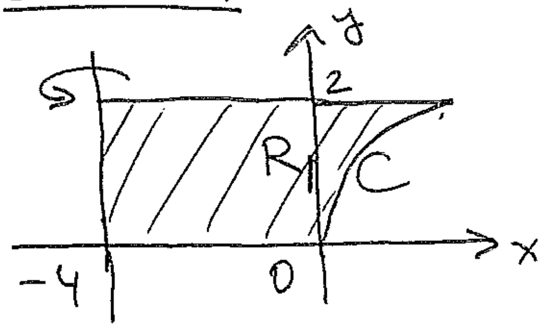
5. b)



C revolved about the line $x=-4$. Volume = ?

$C: y = 3x^{1/3}$ or $x = \dfrac{y^3}{27}$

Solution 1

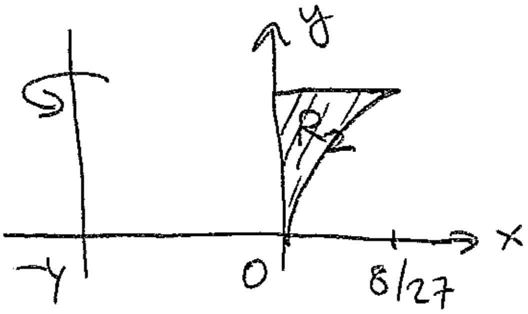


R_1 revolved.

$$V = \pi \int_0^2 \left(\frac{y^3}{27} + 4\right)^2 dy$$

$$= 2\pi \int_{-4}^{0} (x+4)\cdot 2\,dx + 2\pi \int_0^{8/27} (x+4)(2-3x^{1/3})\,dx$$

Solution 2

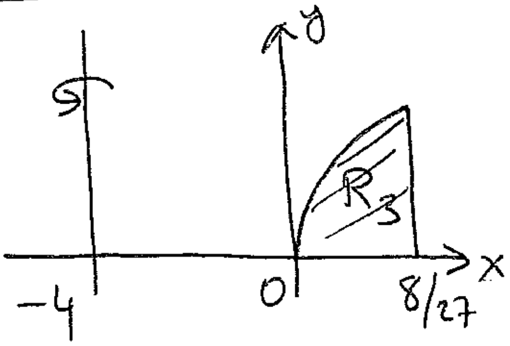


R_2 revolved

$$V = \int_0^{8/27} 2\pi(x+4)(2-3x^{1/3})\,dx$$

$$= \pi \int_0^2 \left(\frac{y^3}{27} + 4\right)^2 dy - \pi\cdot 4^2\cdot 2$$

Solution 3



R_3 revolved

$$V = \int_0^{8/27} 2\pi\,(x+4)\,3x^{1/3}\,dx$$

$$= \pi\left(\frac{8}{27}+4\right)^2\cdot 2 - \pi \int_0^2 \left(\frac{y^3}{27}+4\right)^2 dy$$

Solution 4 C revolved $\Rightarrow$ $V = 0$.