1. Assume f is a differentiable function with an inverse, and also assume that $f(3) = e^2, f'(3) = 7$.

Let $y = f^{-1}\left(e^{x^2+x}\right)$. Find $\frac{dy}{dx}$ at $x = 1$.

Let $u = e^{x^2+x}$. Then $y = f^{-1}(u)$.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{f'(f^{-1}(u))}\frac{du}{dx}, \quad \frac{du}{dx} = e^{x^2+x}\cdot(2x+1)$$

At $x=1$, $\left.\frac{du}{dx}\right|_{x=1} = e^2 \cdot 3$

$x=1 \Rightarrow u = e^2$, $f^{-1}(e^2) = 3$. So $\frac{1}{f'(f^{-1}(u))} = \frac{1}{f'(f^{-1}(e^2))}$

$$= \frac{1}{f'(3)} = \frac{1}{7}.$$

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{1}{7}\cdot e^2 \cdot 3 = \frac{3e^2}{7}.$$

2. Let f be a function so that f' is continuous on the open interval $(3,7)$. Assume that the linearization of $f(x)$ at $x = 4$ is $2x + 5$ and the linearization of $f(x)$ at $x = 5$ is $3x - 5$. Then

a) Find $f(4), f(5), f'(4)$, and $f'(5)$.

b) Show that there exists $c \in (3,7)$ such that $f'(c) = \frac{5}{2}$.

c) Show that there exists $d \in (3,7)$ such that $f'(d) = -2$.

a) $L_1(x) = 2x+5 \rightarrow$ linearization of f at $x=4$

$L_2(x) = 3x-5 \rightarrow$ " " " " $x=5$

So,

$f(4) = L_1(4) = 13$, $f'(4) = L_1'(4) = 2$

$f(5) = L_2(5) = 10$, $f'(5) = L_1'(5) = 3$

b) Let $g(x) = f'(x) - \frac{5}{2}$.

$g(4) = f'(4) - \frac{5}{2} = 2 - \frac{5}{2} < 0$

$g(5) = f'(5) - \frac{5}{2} = 3 - \frac{5}{2} > 0$

$\Big\}$ + g is cont. $\xrightarrow{IVT} \exists c \in (4,5)$ s.t. $g(c) = f'(c) - \frac{5}{2} = 0$.

Since the open interval $(4,5)$ is contained in the open interval $(3,7)$, $c \in (3,7)$.

OR

$f'(4) = 2 < \frac{5}{2} < f'(5) = 3$ & f' is cont. on $(3,7)$ $\xrightarrow{IVT} \exists c \in (4,5)$ s.t. $f'(c) = \frac{5}{2}$

c)

$f' \in C(4,5) \xrightarrow{MVT} \exists c_1 \in (4,5)$ s.t. $f'(c_1) = \frac{f(5)-f(4)}{5-4}$

$= \frac{10-13}{1}$

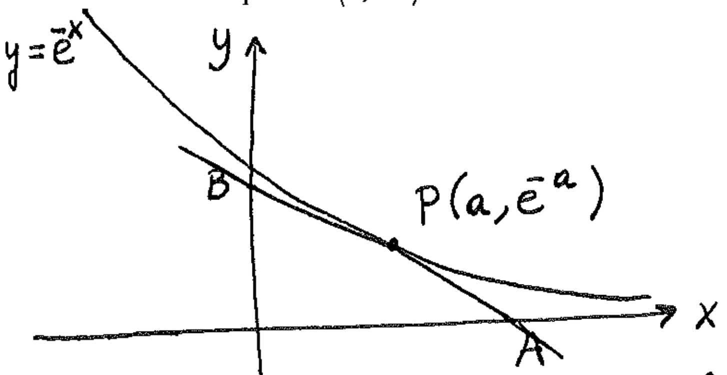
$= -3$

& $f'(4) = 2$

$-3 < -2 < 2 \xrightarrow{IVT} \exists d \in (4, c_1)$ s.t.

$f'(d) = -2$.

3. A particle is moving on the curve $y = e^{-x}$ in such a way that x-coordinate is increasing at a rate of 3 cm/sec. Find the rate of change of the area of the triangle formed by the tangent line to the curve at the location of the particle, the x-axis and the y-axis when the particle is at the point $P(2, e^{-2})$.



Denote the coordinates of the particle by $P(a, e^{-a})$.

$$\frac{dy}{dx} = -e^{-x} \Rightarrow m_{tg.} = -e^{-a}.$$

$$\ell_{tg.}: \quad y - e^{-a} = -e^{-a}(x-a).$$

At A, $y=0$. So $-e^{-a} = -e^{-a}(x-a)$

$$\Rightarrow x = a+1, \text{ i.e., } A(a+1, 0)$$

At B, $x=0$. So $y - e^{-a}(-a) \Rightarrow y = (a+1)e^{-a}$, i.e., $B(0, (a+1)e^{-a})$.

Then $\text{Area} = \frac{1}{2}(a+1)(a+1)e^{-a} = \frac{1}{2}(a+1)^2 e^{-a}$.

$$\frac{d(\text{Area})}{dt} = \frac{1}{2}\left[2(a+1)e^{-a} + (a+1)^2 e^{-a}(-1)\right]\frac{da}{dt}$$

$$= \frac{1}{2}(a+1)e^{-a}\left[\underbrace{2-(a+1)}_{1-a}\right]\frac{da}{dt}$$

$$= \frac{1}{2}(1-a^2)e^{-a}\frac{da}{dt} \quad \text{where } \frac{da}{dt} = 3 \text{ and } a = 2$$

So $\dfrac{d(\text{Area})}{dt} = \dfrac{1}{2}(1-4)e^{-2}\cdot 3 = \dfrac{-9e^{-2}}{2}.$

4. Find an equation of the
a) horizontal tangent line(s),and
b) vertical tangent line(s)
to the curve $x^3 + y^3 - 9xy = 0$.

First note that if a point (a,b) is on the curve x^3+y^3-9xy then (b,a) is also on it. Hence if the curve has a horizontal tangent at (a,b) then it has a vertical tangent at (b,a). Therefore the curve x^3+y^3-9xy doesnot have a horizontal tangent or vertical tangent at $(0,0)$. Suppose the curve x^3+y^3-9xy has a horizontal tangent at (a,b). ~~Then~~ Since we have

$$\boxed{x^3+y^3-9xy=0} \Rightarrow \boxed{3x^2+3y^2\frac{dy}{dx}-9y-9x\cdot\frac{dy}{dx}=0}$$

We get

$$\frac{dy}{dx}=\frac{-x^2+3y}{y^2-3x} \quad \text{when} \quad y^2-3x\neq 0$$

Then $-a^2+3b=0$ and $a^3+b^3-9ab=0$

So $a^3+\frac{a^6}{27}-9a\frac{a^2}{3}=0$ So $a^6=54a^3$

Hence $a=0$ or $a=\sqrt[3]{54}$. But $a\neq 0$ because otherwise $b=0$ and we already know that there is no horizontal tangent line at $(0,0)$.

So $a=\sqrt[3]{54}$ and $b=\frac{\sqrt[3]{54^2}}{3}$

So the only horizontal tangent line is $y=\frac{\sqrt[3]{54^2}}{3}$ and by symmetry the only vertical tangent line is $x=\frac{\sqrt[3]{54^2}}{3}$

5.(1+1+2+2+2+2+2+2+6 pts.) Consider $f(x) = \dfrac{(x+1)^2}{x^2+1}$ .Find:

a) The domain D of f .
b) The x -intercept:
The y -intercept:
c) Asymptote(s):
d) All critical points:
e) All intervals on which f is increasing,decreasing:
f) All local extrema:
g) All intervals on which f is concave up,concave down:
h) All points of inflection:
AND
i) Sketch the graph of $y = f(x)$.

a) $D = \mathbb{R}$

b) $P_0(-1,0)$: x-intercept
$P_1(0,1)$: y-intercept

c) (V.a) None
(o.a) None
(h.a) $\lim\limits_{x \to \mp\infty} f(x) = 1 \Rightarrow y = 1$ is a h.a.

d) $f'(x) = \dfrac{2(1-x^2)}{(1+x^2)^2} = 0 \Rightarrow x = \mp 1$ (c.pts.)

& $f''(x) = \dfrac{4x(x^2-3)}{(1+x^2)^3}$

x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
f'	$-$		$-$	0	$+$		$+$	0	$-$		$-$
f''	$-$	0	$+$		$+$	0	$-$		$-$	0	$+$
f		A		0		1		2		B	

$A = f(-\sqrt{3}) = \dfrac{(1-\sqrt{3})^2}{4} < B = f(\sqrt{3}) = \dfrac{(1+\sqrt{3})^2}{4}$

e) f is decreasing on the intervals $(-\infty,-1)$ and $(1,\infty)$
f is increasing on the interval $(-1,1)$

f) $f(-1) = 0$ is a local min.
$f(1) = 2$ " " " max.

g) the curve is concave down on the intervals $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$, and

the curve is concave up on the intervals $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$.

h) The second derivative changes sign at the points $x = -\sqrt{3}$, $x = 0$, and $x = \sqrt{3}$. Thus each point is a point of inflection.