1) (30 points)Answer the questions from a) to g) for the function $f(x) = \dfrac{x^3}{(x-1)^2}$.

a) (1 point) Find the domain of $f(x) = \dfrac{x^3}{(x-1)^2}$.

$\text{Dom } f(x) = \mathbb{R} \setminus \{1\}$

b) (1 point) Find the intercept points of $f(x) = \dfrac{x^3}{(x-1)^2}$.

$x = 0 \Rightarrow y = 0 \Rightarrow (0,0)$

c) (6 points) Find all asymptotes (horizontal, vertical, oblique) of $f(x) = \dfrac{x^3}{(x-1)^2}$ if they exist. If they do not exist, explain why not.

		0		1	
x^3	$-$	0	$+$	|	$+$
$(x-1)^2$	$+$		$+$	|	$+$
f	$-$		$+$	|	$+$

$\left.\begin{array}{l} \lim\limits_{x\to-\infty} f(x) = -\infty \\ \lim\limits_{x\to+\infty} f(x) = +\infty \end{array}\right\}$ No horizontal asymptote

$x = 1$ vertical asymptote

$\lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1^+} f(x) = +\infty$

$$\begin{array}{l|l} x^3 & x^2-2x+1 \\ \hline -x^3+2x^2-x & x+2 \\ \hline 2x^2-x & \\ -2x^2+4x-2 & \\ \hline 3x-2 & \end{array}$$

$\dfrac{x^3}{x^2-2x+1} = x+2+\dfrac{3x-2}{(x-1)^2} \Rightarrow y = x+2$ is the oblique asympt.

d) (6 points) Find all critical points, interval of increase and decrease, maximum-minimum values of $f(x) = \dfrac{x^3}{(x-1)^2}$. Explain your steps.

$$f'(x) = \frac{3x^2(x-1)^{\cancel{2}\,1} - x^3 \cdot 2\cancel{(x-1)}}{(x-1)^{\cancel{4}\,3}} = \frac{3x^2(x-1)-2x^3}{(x-1)^3} = \frac{3x^3-3x^2-2x^3}{(x-1)^3} = \frac{x^3-3x^2}{(x-1)^3}$$

$= \dfrac{x^2(x-3)}{(x-1)^3} = 0 \Rightarrow x=0$ & $x=3$ are the only critical points

because $f'(x)$ is undefined at $x=1$ but $x = 1 \notin \text{Dom} f(x)$.

x	$-\infty$	1		3	
$x-3$	$-$	|	$-$	0	$+$
$(x-1)^3$	$-$	0	$+$		$+$
$f'(x)$	$+$ ↗	|	$-$ ↘	l min. $\frac{27}{4}$	$+$ ↗

$f(0) = 0$

$f(3) = \dfrac{27}{4}$

f is increasing on $(-\infty, 1) \cup (3, +\infty)$

f is decreasing on $(1, 3)$.

e) (4 points) Find intervals of concavity and inflection points of $f(x) = \frac{x^3}{(x-1)^2}$ if any. Explain your steps.

$$f''(x) = \frac{(3x^2-6x)(x-1)^{\cancel{3}\,1} - (x^3-3x^2)\cdot 3(x-1)^{\cancel{2}}}{(x-1)^{\cancel{6}\,4}} = \frac{(3x^2-6x)(x-1) - 3(x^3-3x^2)}{(x-1)^4}$$

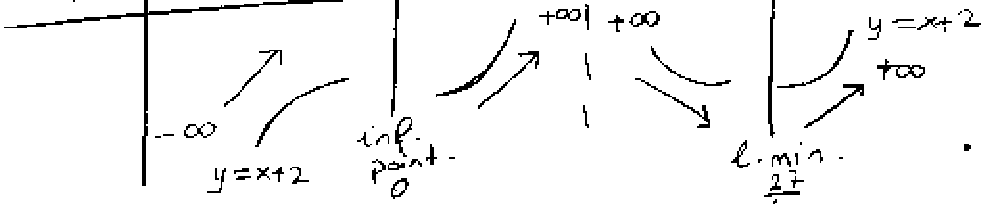
$$= \frac{\cancel{3x^3} - 3x^2 - \cancel{6x^2} + 6x - \cancel{3x^3} + \cancel{9x^2}}{(x-1)^4} = \frac{6x}{(x-1)^4} = 0 \Rightarrow x = 0$$

x		0	
$6x$	—	0	+
f''	∩		∪

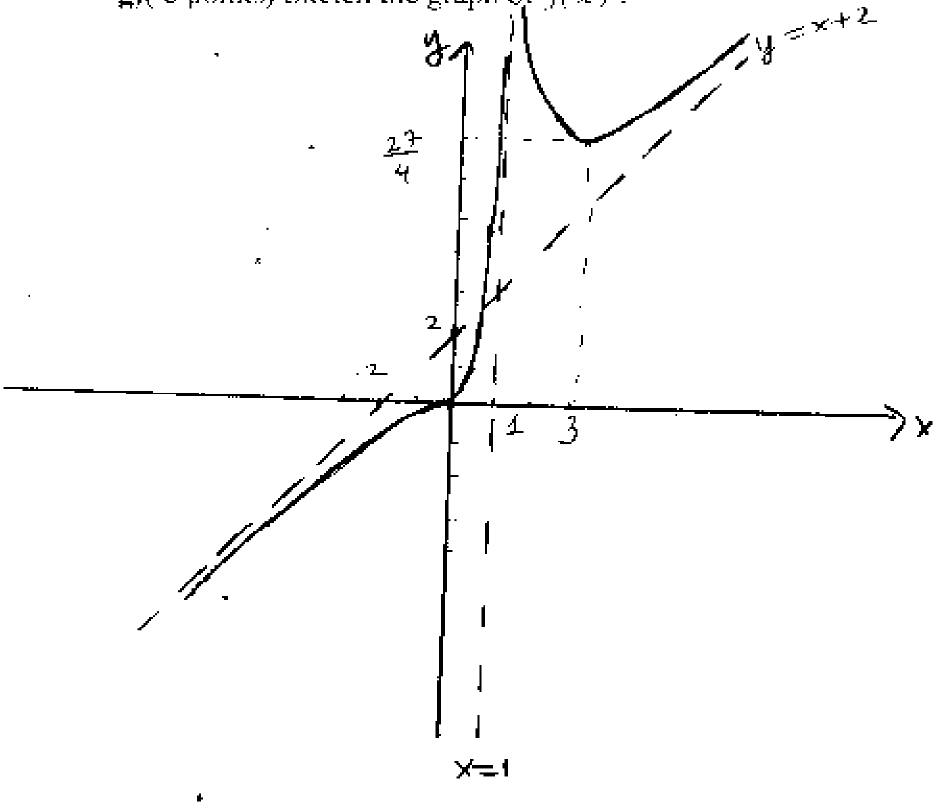
$x=0$ is an inflection point.

f) (4 points) Make a **complete** table.

x	$-\infty$	0	1	3	$+\infty$
f'	+	+	− −	+	
f''	—	+	+	+	
f	↗	↗	↘	↗	



g)(8 points) Sketch the graph of $f(x)$.



2) **a)** (5 points) State the Mean Value Theorem.

If $f(x)$ is continuous on $[a,b]$ and diff'ble on (a,b) then there exists a point $c \in (a,b)$ s.t. $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

b) (15 points) Let $f(x)$ be a differentiable function such that $f'(x) = \dfrac{1}{1+x^2}$.

Show that for all $0<a<b$, $\dfrac{b-a}{1+b^2} < f(b)-f(a) < \dfrac{b-a}{1+a^2}$ by using the **Mean Value Theorem.**

Since $f(x)$ is diff'ble and $0<a<c<b$

then $f'(c) = \dfrac{f(b)-f(a)}{b-a} \Rightarrow \boxed{(b-a)\, f'(c) = f(b)-f(a)}$

$$0<a<c<b \Rightarrow a^2 < c^2 < b^2$$

$$\Rightarrow 1+a^2 < 1+c^2 < 1+b^2$$

$$\Rightarrow \frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$$

$$\Rightarrow \frac{b-a}{1+b^2} < \underbrace{\frac{b-a}{1+c^2}}_{= f'(c)(b-a)} < \frac{b-a}{1+a^2} \quad \text{since } b-a>0$$

$$\Rightarrow \frac{b-a}{1+b^2} < \underbrace{f'(c)\,(b-a)}_{= f(b)-f(a)} < \frac{b-a}{1+a^2}$$

$$\Rightarrow \boxed{\frac{b-a}{1+b^2} < f(b)-f(a) < \frac{b-a}{1+a^2}}$$

3) a) (5 points) Evaluate $\int \frac{x^3}{\sqrt{x^2+25}}\,dx$.

$u = x^2+25 \Rightarrow du = 2x\,dx \Rightarrow \frac{du}{2} = x\,dx$

$\Downarrow$

$\boxed{x^2 = u-25}$

$$I = \int \frac{x^2 \cdot x\,dx}{\sqrt{x^2+25}} = \frac{1}{2}\int \frac{(u-25)\,du}{\sqrt{u}} = \frac{1}{2}\int\left(\frac{u}{\sqrt{u}} - \frac{25}{\sqrt{u}}\right)du$$

$$= \frac{1}{2}\left[\int \sqrt{u}\,du - 25\int \frac{du}{\sqrt{u}}\right] = \frac{1}{2}\,\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} - 25\cdot 2\sqrt{u} + C$$

$$= \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}\,u^{3/2} - 25\sqrt{u} + C = \frac{1}{3}\cdot(x^2+25)^{3/2} - 25(x^2+25) + C.$$

b) (5 points) Evaluate $\int\sqrt{\sin x}\,\cos^3 x\,dx$.

$$\int\sqrt{\sin x}\cdot\cos^3 x\,dx = \int\sqrt{\sin x}\,\cos^2 x\cdot\cos x\,dx = \int\sqrt{\sin x}\cdot(1-\sin^2 x)\cdot\cos x\,dx$$

$\boxed{u = \sin x \Rightarrow du = \cos x\,dx}$

$$= \int\sqrt{u}\cdot(1-u^2)\,du = \int\left(\sqrt{u} - u^{5/2}\right)du = \frac{2}{3}u^{3/2} - \frac{u^{\frac{5}{2}+1}}{\frac{5}{2}+1} + C$$

$$= \frac{2}{3}u^{3/2} - \frac{2}{7}u^{7/2} + C = \frac{2}{3}(\sin x)^{3/2} - \frac{2}{7}(\sin x)^{7/2} + C.$$

c) (5 points) $\int (1+x^2)^{-3}\cdot x\,dx = \frac{1}{2}\int u^{-3}\,du = \frac{1}{2}\,\frac{u^{-3+1}}{-3+1} + C$

$1+x^2 = u \Rightarrow 2x\,dx = du \Rightarrow x\,dx = \frac{du}{2}$

$$= \frac{1}{2}\,\frac{u^{-2}}{-2} + C = \frac{(1+x^2)^{-2}}{-4} + C.$$

4) Let $g(x)$ be continuous everywhere, $g(1)=1$, $\int_0^1 g(t)dt = 2$.

Let $f(x)=\frac{1}{2}\int_0^x (x-t)^2 g(t)dt$.

a) (10 points) Show that $f'(x) = x\int_0^x g(t)dt - \int_0^x tg(t)dt$.

$$f(x)=\frac{1}{2}\int_0^x (x^2-2xt+t^2)g(t)dt = \frac{1}{2}\left[\int_0^x x^2 g(t)dt - \int_0^x 2xt\, g(t)dt + \int_0^x t^2 g(t)dt\right]$$

$$=\frac{1}{2}\left[x^2\int_0^x g(t)dt - 2x\int_0^x t g(t)dt + \int_0^x t^2 g(t)dt\right]$$

$$f'(x)=\frac{1}{2}\left[2x\int_0^x g(t)dt + \cancel{x^2 g(x)} - 2\int_0^x t g(t)dt - \cancel{2x^2 g(x)} + \cancel{x^2 g(x)}\right]$$

$$= x\int_0^x g(t)dt - \int_0^x t g(t)dt$$

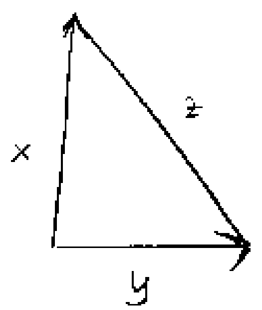
b) (10 points) Compute $f''(1)$ and $f'''(1)$.

$$f''(x) = \int_0^x g(t)dt + \cancel{x\cdot g(x)} - \cancel{x\cdot g(x)} = \int_0^x g(t)dt$$

$$f''(1) = \int_0^1 g(t)dt = 2.$$

$$f'''(x) = g(x) \Rightarrow f'''(1) = g(1) = \underline{\underline{1}}$$

5)(15 points) At time $t = 0$, an insect starts flying along a vertical line at a rate of 2 ft/min. Three minutes later, a second insect starts from the same position, flying in a direction perpendicular to that of the first and at a speed of 4ft/min. How fast is the distance between them changing once the first insect has traveled 10ft?



$\frac{dx}{dt} = 2 \text{ ft/min}$

$\frac{dy}{dt} = 4 \text{ ft/min}$

$\frac{dz}{dt} = ?$

when $x = 10$ ft $\Rightarrow t = 5$ min

After the 2 min. the 2^{nd} insect takes $8 ft = y$

$x = 10$ & $y = 8 \Rightarrow x^2 + y^2 = z^2 \Rightarrow 100 + 64 = z^2 \Rightarrow z = \sqrt{164}$
$= 2\sqrt{41}$

$\Rightarrow \cancel{2}x \frac{dx}{dt} + \cancel{2}y \frac{dy}{dt} = \cancel{2}z \cdot \frac{dz}{dt}$

$\Rightarrow 10 \cdot 2 + 8 \cdot 4 = 2\sqrt{41} \frac{dz}{dt}$

$\Rightarrow \frac{20 + 32}{2\sqrt{41}} = \frac{dz}{dt}$

$\Rightarrow \frac{\cancel{52}^{26}}{\cancel{2}\sqrt{41}} = \frac{26}{\sqrt{41}} = \frac{26 \cdot \sqrt{41}}{41} = \frac{dz}{dt}$