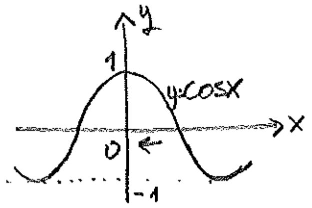
Math. 111 (2009-10, MT1, Fall)

Questions + Answers

Remember: You are not allowed to use the L´ Hopital´s rule.

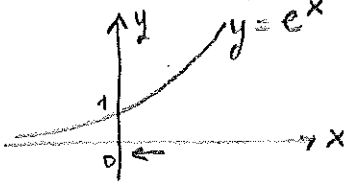
1a) Given that $\lim\limits_{x\to 1^-} f(x) = 3$, $\lim\limits_{x\to 1^+} f(x) = 5$, evaluate $\lim\limits_{x\to 0^+} \dfrac{f(\cos x)}{f(e^x)}$.



$$\lim_{x\to 0^+} \frac{f(\cos x)}{f(e^x)} = \frac{\lim\limits_{x\to 0^+} f(\cos x)}{\lim\limits_{x\to 0^+} f(e^x)} = \frac{5}{3}.$$

As $x\to 0^+$, $\cos x \to 1^-$

+ as $x\to 0^+$, $e^x \to 1^+$



b) Evaluate $\lim\limits_{x\to 0} \dfrac{(\tan x^2)(\sin 2x)}{x^3}$ without using the L'Hopital's Rule.

$$\lim_{x\to 0} \frac{(\tan x^2)(\sin 2x)}{x^3} = \lim_{x\to 0} \frac{(\sin x^2)(\sin 2x)}{x^2 \cdot \frac{2x}{2} \cdot \cos x^2}$$

$$= \underbrace{\left(\lim_{x\to 0} \frac{\sin x^2}{x^2}\right)}_{1} \underbrace{\left(\lim_{x\to 0} \frac{\sin 2x}{2x}\right)}_{1} \underbrace{\left(\lim_{x\to 0} \frac{2}{\cos x^2}\right)}_{2}$$

$$= 1 \cdot 1 \cdot 2$$

$$= 2$$

Remember:

$$\lim_{t\to 0} \frac{\sin t}{t} = 1$$

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2a)Find $\lim_{x\to 1} f(x)$ if $|f(x)-3| \le \frac{|x-1|}{2}$ for all $x \in (-3,4)$. Explain your answer.

$$\lim_{x\to 1}\left[-\frac{|x-1|}{2} \le f(x)-3 \le \frac{|x-1|}{2}\right]$$

$$\underbrace{-\frac{|x-1|}{2}}_{\to 0} \qquad \underbrace{\frac{|x-1|}{2}}_{\to 0}$$

$\xRightarrow[\text{Thm.}]{\text{Sandwich}} \lim_{x\to 1}(f(x)-3)=0$ or $\lim_{x\to 1} f(x)=3$.

OR

$$\lim_{x\to 1}\left[0 \le |f(x)-3| \le \frac{|x-1|}{2}\right]$$

$$0 \to 0 \qquad \underbrace{\frac{|x-1|}{2}}_{\to 0}$$

$\xRightarrow[\text{Thm.}]{\text{Sandwich}} \lim_{x\to 1}|f(x)-3|=0 \Longrightarrow \lim_{x\to 1} f(x)=3$ since $-|f(x)-3| \le f(x)-3 \le |f(x)-3|$ and $-|f(x)-3|$ and $|f(x)-3|$ have limit 0 as $x\to 1$.

b)If $f(x) = x^3 - x^2 + x$, show that there is a number c such that $f(c) = 10$.

$\left.\begin{array}{l} f \text{ is cont. everywhere, and} \\ f(2)=6 \text{ and } f(3)=21 \end{array}\right\} \xRightarrow{\text{IVT}} \exists c \in (2,3)$ s.t. $f(c)=10$

$$\underset{f(2)}{\overset{\|}{6}} < \underset{f(c)}{\overset{\|}{10}} < \underset{f(3)}{\overset{\|}{21}}$$

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3a) Find all possible values of m and n such that

$$f(x)=\begin{cases}\dfrac{x-|mx|}{x}, & x<0\\ n, & x=0\\ \dfrac{x^2\left(e^{\frac{-1}{x}}+2\right)}{1-\cos x}, & x>0.\end{cases}$$

is continuous at $x=0$.

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{x-|mx|}{x}=\lim_{x\to 0^-}\frac{x+|m|x}{x}=\lim_{x\to 0^-}\frac{x(1+|m|)}{x}=1+|m|$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{x^2(e^{-1/x}+2)}{1-\cos x}\cdot\frac{1+\cos x}{1+\cos x} \quad \text{where } 1-\cos^2 x=\sin^2 x$$

$$=\underbrace{\left(\lim_{x\to 0^+}\frac{x^2}{\sin^2 x}\right)}_{1}\underbrace{\left(\lim_{x\to 0^+}(e^{-1/x}+2)\right)}_{2}\underbrace{\left(\lim_{x\to 0}(1+\cos x)\right)}_{2}=4$$

$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=4 \iff 1+|m|=4 \iff m=3$ or $m=-3$

So, $\lim_{x\to 0} f(x)=4 \iff m=3$ or $m=-3$.

If $n=4$, then $f(0)=4=\lim_{x\to 0} f(x)$.

$\therefore$ $f(x)$ is cont. at $x_0=0$ if $m=3$ or $m=-3$ and $n=4$.

b) Find two functions f and g such that $\lim_{x\to -1}(f(x)g(x))$ exists but neither $\lim_{x\to -1} f(x)$ nor $\lim_{x\to -1} g(x)$ exists.

Neither $\lim_{x\to -1}\frac{|x+1|}{x+1}$ nor $\lim_{x\to -1}\frac{x+1}{|x+1|}$ exists but

$$\lim_{x\to -1}\left(\frac{|x+1|}{x+1}\cdot\frac{x+1}{|x+1|}\right)=\lim_{x\to -1} 1=1.$$

Remark: There are several pairs of fcns. f and g with the above property.

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4.Suppose f is a function that satisfies the equation $f(x+y)=f(x)+f(y)+x^2y+xy^2$ for all real numbers x and y. Suppose also that $\lim_{x\to 0}\frac{f(x)}{x}=1$.

a)Find $f(0)$.
b)Find $f'(0)$.
c)Find $f'(x)$.

a) $f(0+0)=f(0)+f(0) \Rightarrow f(0)=0$.

b) $f'(0)=\lim_{h\to 0}\frac{f(0+h)-\overset{0}{f(0)}}{\underset{0}{}h}$

$=\lim_{h\to 0}\frac{\overset{0}{f(0)}+f(h)}{h}$

$=\lim_{h\to 0}\frac{f(h)}{h}$ since $\lim_{h\to 0}\frac{f(x)}{x}=1$

$=1$

c) $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$=\lim_{h\to 0}\frac{\cancel{f(x)}+f(h)+x^2h+xh^2-\cancel{f(x)}}{h}$

$=\lim_{h\to 0}\frac{f(h)+h(x^2+xh)}{h}$

$=\underbrace{\left(\lim_{h\to 0}\frac{f(h)}{h}\right)}_{1}+\left(\underbrace{\lim_{h\to 0}\frac{h}{h}}_{1}\right)\left(\underbrace{\lim_{h\to 0}(x^2+xh)}_{x^2}\right)$

$=x^2+1$.

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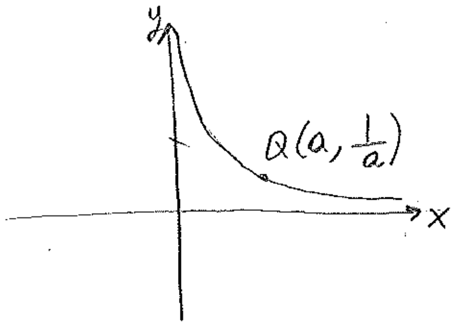
5a) Write an equation of the tangent line to the curve $y = \frac{1}{x}$ passing through the point $P(0,1)$.



$P(0,1)$ doesn't lie on the curve $y=\frac{1}{x}$

Take a point $Q(a,\frac{1}{a})$, $a\neq 0$ which lies on the curve $y=\frac{1}{x}$. Then the slope m of the tg. line to the graph of $y=\frac{1}{x}$ at $Q(a,\frac{1}{a})$ is

$$m = \frac{d}{dx}\left(\frac{1}{x}\right)\Big|_{x=a} = -\frac{1}{a^2}.$$

And

$y-\frac{1}{a} = -\frac{1}{a^2}(x-a)$ is an eqn. of the tg. line to the curve $y=\frac{1}{x}$ at the point $Q(a,\frac{1}{a})$.

$1-\frac{1}{a} = -\frac{1}{a^2}(-a)$ or $a=2$ since the tg. line passes through P. and $Q(2,\frac{1}{2})$

Then $y-\frac{1}{2} = -\frac{1}{4}(x-2)$ or $y = 1-\frac{x}{4}$ is an eqn. of the tg. line to the graph of $y=\frac{1}{x}$ passing through $P(0,1)$.

b) Calculate $\lim_{x\to 2}\frac{f(x^2+5)-f(9)}{x-2}$ if $f'(x) = e^{-x^2}$.

$$\lim_{x\to 2}\frac{f(x^2+5)-f(9)}{x-2} = \lim_{u\to 0}\frac{f((u+2)^2+5)-f(9)}{u}$$

Set $u = x-2$.
As $x\to 2$, $u\to 0$

$$= \lim_{u\to 0}\frac{f(9+4u+u^2)-f(9)}{u}\cdot\frac{4u+u^2}{4u+u^2}$$

$$= \left(\lim_{u\to 0}\frac{f(9+4u+u^2)-f(9)}{4u+u^2}\right)\left(\lim_{u\to 0}\frac{4u+u^2}{u}\right)$$

Set $h = 4u+u^2$.
As $u\to 0$, $h\to 0$

$$= \underbrace{\left(\lim_{h\to 0}\frac{f(9+h)-f(9)}{h}\right)}_{f'(9)}\underbrace{\left(\lim_{h\to 0}\frac{u(4+u)}{u}\right)}_{4}$$

$$= 4e^{-81}.$$