Math. 111 (2009-10, Fall)

MT1, Questions + Answers

(10+10 pts.) 1) Evaluate

a) $\lim\limits_{x\to 0}\left(e^x - 3x\right)^{2/x}$: $[1^\infty]$

$$\lim_{x\to 0}(e^x-3x)^{2/x} = \lim_{x\to 0} e^{\ln(e^x-3x)^{2/x}}$$

$$= \lim_{x\to 0} e^{\frac{2\ln(e^x-3x)}{x}}$$

continuity of exp.

$$= e^{\left(\lim\limits_{x\to 0}\frac{2\ln(e^x-3x)}{x}\right)=L}$$

$$= e^L$$

where $L = \lim\limits_{x\to 0}\dfrac{2\ln(e^x-3x)}{x}$: $\left[\frac{0}{0}\right]$

$$\overset{L'H}{=} \lim_{x\to 0}\frac{\frac{2}{e^x-3x}(e^x-3)}{1}$$

$$= \frac{2(1-3)}{1-0} = -4.$$ So $\lim\limits_{x\to 0}(e^x-3x)^{2/x} = e^{-4}$.

b) $\lim\limits_{x\to 0}\dfrac{\int_0^{x^2}\cos\left(\sqrt{t}\right)dt}{x\sin 3x}$: $\left[\frac{0}{0}\right]$

$$\lim_{x\to 0}\frac{\int_0^{x^2}\cos(\sqrt{t})\,dt}{x\sin 3x} \overset{L'H}{=} \lim_{x\to 0}\frac{\cos(\sqrt{x^2})\cdot 2x}{\sin 3x + 3x\cos 3x}$$

$$= \underbrace{\left(\lim_{x\to 0} 2\cos|x|\right)}_{2}\left(\lim_{x\to 0}\frac{1}{3\underbrace{\frac{\sin 3x}{3x}}_{=1} + \frac{3\cos 3x}{1}}\right)$$

$$= \frac{2}{6} = \frac{1}{3}.$$

(12+8 pts.)2) a) Let a function $f:\Re \to \Re$ have the following properties

- $|f(u) - f(v)| \le 3|u - v|^2$ for all $u, v \in \Re$, and
- $f(1) = -8$.

i) Find $f'(x)$.

ii) Find $f(x)$.

i) $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

where $|f(x+h) - f(x)| \le 3|x+h-x|^2 = 3|h|^2$

Then $\left|\frac{f(x+h) - f(x)}{h}\right| \le \frac{3|h|^2}{|h|} = 3|h|$

$\Rightarrow \lim_{h\to 0}\left(-3|h| \le \frac{f(x+h)-f(x)}{h} \le 3|h|\right)$

Sandwich thm. $\Rightarrow \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = 0$.

That is $f'(x) = 0 \quad \forall x$.

ii) $f(x) = c \in \mathbb{R}$. $f(1) = -8 \Rightarrow c = -8$. So $f(x) = -8$.

b) $\frac{d}{dx}\left[(e^x - 3x)^{2/x}\right]$

Let $y = (e^x - 3x)^{2/x}$. Then $\ln y = \frac{2}{x}\ln(e^x - 3x)$.

$\frac{1}{y}y' = -\frac{2}{x^2}\ln(e^x - 3x) + \frac{2}{x}\frac{e^x - 3}{e^x - 3x}$

$\Rightarrow y' = (e^x - 3x)^{2/x}\left(-\frac{2}{x^2}\ln(e^x - 3x) + \frac{2}{x}\frac{e^x - 3}{e^x - 3x}\right)$.

(8+6+6 pts.)3) Evaluate

a) $\lim_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n}\left(e^{k/n}+\sin\left(\frac{n+2k}{n}\right)\right)\right] = \lim_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n}\left(e^{k/n}+\sin\left(1+2\frac{k}{n}\right)\right)\right]$

$$= \int_0^1 \left(e^x + \sin(1+2x)\right)dx$$

$$= \left(e^x - \frac{\cos(1+2x)}{2}\right)\Bigg|_0^1$$

$$= e - \frac{\cos 3}{2} - e^0 + \frac{\cos 1}{2}$$

$$= e - 1 - \frac{\cos 3}{2} + \frac{\cos 1}{2}.$$

b) $\int e^{(x+e^x)}dx = \int e^x . e^{(e^x)} dx$

$\underline{\underline{u = e^x, du = e^x dx}}$ $\int e^u du$

$$= e^u + c$$

$$= e^{(e^x)} + c.$$

c) $\int_0^{\frac{\pi}{4}} \sqrt{\tan x}\sec^2 x dx = 2\int_0^1 u^2 du = \frac{2}{3}u^3\Big|_{u=0}^{1} = \frac{2}{3}$

$u = \sqrt{\tan x}$

$du = \frac{\sec^2 x}{2\sqrt{\tan x}} dx$

$\Rightarrow 2u\, du = \sec^2 x\, dx$

$x = 0 \Rightarrow u = 0$

$x = \frac{\pi}{4} \Rightarrow u = 1$

<u>OR</u> $w = \tan x \Rightarrow dw = \sec^2 x dx$

Then

$$\int_0^{\pi/4} \sqrt{\tan x}\sec^2 x dx = \int_0^1 \sqrt{w}\, dw$$

$$= \frac{2}{3} w^{3/2}\Big|_0^1$$

$$= \frac{2}{3}.$$

(10+10 pts.)4) a) Find the value of m such that

$$f(x)=\begin{cases}\dfrac{1}{x-\sin x}\displaystyle\int_0^x \frac{t^2\,dt}{\sqrt{4+e^t}} & , \quad x\neq 0\\ m & , \quad x=0\end{cases}$$

is continuous at $x=0$.

For continuity at $x=0$, we must have

$$m=\lim_{x\to 0}\frac{\int_0^x \frac{t^2dt}{\sqrt{4+e^t}}}{x-\sin x} : \left[\frac{0}{0}\right]$$

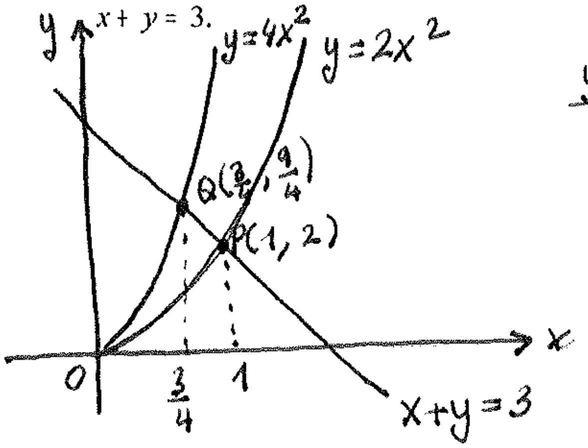
$$\overset{L'H}{=}\lim_{x\to 0}\frac{\frac{x^2}{\sqrt{4+e^x}}}{1-\cos x}=\left(\lim_{x\to 0}\frac{1}{\sqrt{4+e^x}}\right)\left(\lim_{x\to 0}\frac{x^2}{1-\cos x}\right)$$

$$=\frac{2}{\sqrt5} \quad \text{where} \quad \lim_{x\to 0}\frac{x^2}{1-\cos x}\overset{L'H}{=}\lim_{x\to 0}\frac{2x}{\sin x}=2$$

OR

$$=\lim_{x\to 0}\frac{x^2(1+\cos x)}{\sin^2 x}=2.$$

b) Find the area of the region in the first quadrant bounded by $y=2x^2$, $y=4x^2$, $x+y=3$.



$\underline{y=2x^2 \cap y+x=3}$:

$2x^2=3-x \Rightarrow 2x^2+x-3=0$

$\Rightarrow x_{1,2}=\dfrac{-1\mp\sqrt{1-4(2)(-3)}}{4}$

$=\dfrac{-1\mp5}{4}$ ↗ 1, ↘ $\cancel{-\frac{3}{2}}$

$x=1 \Rightarrow y=2$

so $P(1,2)$.

$\underline{y=4x^2 \cap x+y=3}$: $4x^2=3-x \Rightarrow 4x^2+x-3=0$

$\Rightarrow x_{1,2}=\dfrac{-1\mp\sqrt{1-4(4)(-3)}}{8}$

$=\dfrac{-1\mp7}{8}$ ↗ $\cancel{-1}$, ↘ $\frac{3}{4}$

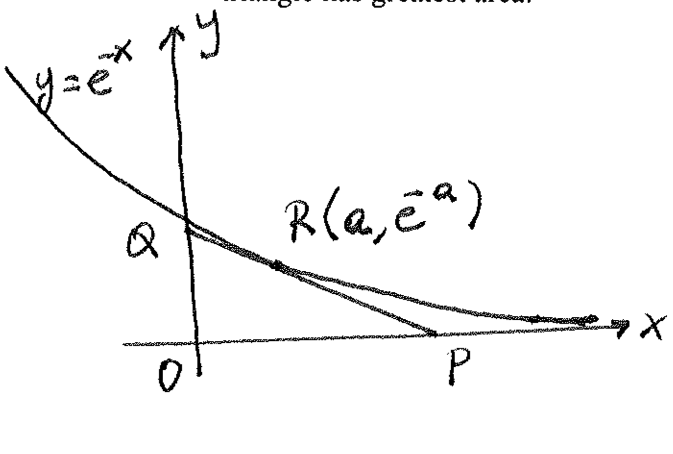
$x=\frac34 \Rightarrow y=\frac94$

$$A=\int_0^{3/4}(4x^2-2x^2)\,dx+\int_{3/4}^{1}\left((3-x)-(2x^2)\right)dx = 41/96$$

OR

$$A=\int_0^{2}\left(\sqrt{\frac{y}{2}}-\frac{\sqrt{y}}{4}\right)dy+\int_2^{9/4}\left((3-y)-\frac{\sqrt{y}}{2}\right)dy=41/96$$

(20 pts.)5) A triangle is formed in the first quadrant using the x - *axis*, y - *axis*, and a tangent line to the graph of the function $y = e^{-x}$ at the point $P(a,b)$, $a \geq 0$. Find a if the triangle has greatest area.



$$y = e^{-x} \Rightarrow y'\Big|_{R(a,e^{-a})} = -e^{-a}$$

tg. line: $y - e^{-a} = -e^{-a}(x-a)$

At P, $y=0$, $-\cancel{e^{-a}} = -\cancel{e^{-a}}(x-a)$

So $x = 1+a$ and

$P(1+a, 0)$

At Q, $x=0$, $y - e^{-a} = -e^{-a}(-a)$

or $y = e^{-a}(a+1)$. So $Q(0, e^{-a}(1+a))$.

$A = \frac{1}{2}(a+1)^2 e^{-a}$, $a \geq 0$

$\frac{dA}{da} = (a+1)e^{-a} - \frac{1}{2}(a+1)^2 e^{-a} = 0 \Rightarrow \underset{\neq 0}{e^{-a}}(a+1)\left(1 - \frac{a+1}{2}\right) = 0$

$\Rightarrow a = -1 \not\geq 0$

or $a = 1$.

a	0		1	
$A'(a)$		$+$	0	$-$
A	$\frac{1}{2}$	$\nearrow$	$\frac{2}{e}$	$\searrow$

$A(0) = \frac{1}{2}$

$A(1) = \frac{2}{e}$

& $\lim_{a \to \infty} A(a) = 0$

$\Rightarrow$ So Area is largest at $a = 1$.