Math.240
HW I
Due:October 14, 2005
Bring the homework to the lecture on the due date.

1.Show that $\phi(x)=\sin x-\cos x$ is a solution to the IVP

$$y''+y=0; \quad y(0)=-1 \ , \ y'(0)=1.$$

$\phi''+\phi=-\sin x+\cos x+\sin x-\cos x=0$,and $\phi(0)=-1$ and $\phi'(0)=1$.
Therefore ϕ is a soln. of the IVP.

2.a)Determine whether the Existence and Uniqueness Theorem implies that the IVP

$$y'=(x-3)(y+1)^{\frac{2}{3}} \ ; \ y(1)=-1.$$

The IVP has a soln. since $f(x,y)=(x-3)(y+1)^{\frac{2}{3}}$ is cont. everywhere but the Existence and Uniqueness Thm. says nothing about uniqueness of its soln.(Why?)

b)Solve the IVP.

$(y+1)^{-\frac{2}{3}}dy=(x-3)dx \quad if \quad y\neq -1$

$\Rightarrow \ y=-1+\left(\frac{x^2}{3}-3x+c\right)^3 \ , \ c\in\mathbb{R}$

$y(1)=-1\Rightarrow c=\frac{5}{2}$

$\therefore y(x)=-1+\left(\frac{x^2}{2}-3x+\frac{5}{2}\right)^3$

On the other hand, $y=-1$ satisfies the IVP.Hence, $y=-1+\left(\frac{x^2}{2}-3x+\frac{5}{2}\right) \ and \ y=-1$ are the solns.

3.Solve $(x+x^6)dy-ydx=0$ by finding an integrating factor of the form $\mu(x,y)=x^{\alpha}y^{\beta}$.

$x^{\alpha}y^{\beta}(x+x^6)dy-x^{\alpha}y^{\beta+1}dx=0 \quad (E)$

Then, $\frac{\partial}{\partial x}\left(y^{\beta}\left(x^{\alpha+1}+x^{\alpha+6}\right)\right)=\frac{\partial}{\partial y}\left(-x^{\alpha}y^{\beta+1}\right) \ \Leftrightarrow \alpha=-6 \ and \ \beta=4.$

Now,let's find $\psi(x,y)$ s.t.

$\psi_x(x,y)=-x^{-6}y^5$ ...... (A)
and
$\psi_y(x,y)=x^{-6}y^4(x+x^6)$ .... (B)

(A) $\Rightarrow \psi(x,y)=\dfrac{x^{-5}y^{5}}{5}+h(y)$

(B) $\Rightarrow h'(y)=y^{4}$

$\Rightarrow h(y)=y^{5}+C, C\in\mathbb{R}$

$\therefore \psi(x,y)=\dfrac{x^{-5}y^{5}}{5}+\dfrac{y^{5}}{5}+C$ and $y^{5}(1+x^{5})=Ex^{5}$ is ageneral soln.of it.

4.Find the continuous solution to the IVP

$$y'+p(x)y=x \;\; ; \;\; y(0)=1,$$

where

$$p(x)=\begin{cases} 1 & if \quad x\in[0,2] \\ 3 & if \quad x\in(2,\infty). \end{cases}$$

$0\le x\le 2\Rightarrow y'+y=x, y(0)=1$

We know that any nonzero constant multiple of e^{x} can be taken as an integrating factor and the general soln. to the IVP can be written as

$y(x)=x-1+2e^{-x} \quad if \quad x\in[0,2]$

$x>2\Rightarrow y'+3y=x$

and $\dfrac{d}{dx}\left(e^{3x}y\right)=xe^{3x}$.

This suggests that $y(x)=\dfrac{x}{3}-\dfrac{1}{9}+ce^{-3x} \quad if \quad x>2.$

$\lim\limits_{x\to 2^-} y=\lim\limits_{x\to 2^+} y=y(2)$ (for the continuity of the soln.)

$\Rightarrow c=e^{-6}\left(\dfrac{4}{9}+2e^{-2}\right) \quad and$

$$y(x)=\begin{cases} x-1+2e^{-x} & if \quad x\in[0,2] \\ \dfrac{x}{3}-\dfrac{1}{9}+e^{-6}\left(\dfrac{4}{9}+2e^{-2}\right)e^{-3x} & if \quad x\in(2,\infty) \end{cases}$$

which is cont. but not differentiable at $x=2.$

5.Use the substitution $y=vx^{2}$ to solve

$$y'=\frac{2y}{x}+x\cos\left(\frac{y}{x^{2}}\right).$$

Let $y=vx^{2}$ then $y'=2xv+x^{2}v'$, and the eqn. is transformed into $x^{2}v'=x\cos v$ which is separable . After separating variables and using integration , we obtain

$\sec v+\tan v=Cx$ and using the back-substitution $y=vx^{2}$

$$\sec\frac{y}{x^2}+\tan\frac{y}{x^2}=Cx, C\in\mathbb{R}.$$