1) a)(8 pts.) Find the solution of the IVP

$$y^{(4)}+2y'''-5y''+11y'+7y=0 \quad , \quad y(1)=y'(1)=y''(1)=y'''(1)=0.$$

$y(x)=0$ satisfies the given fourth-order,linear,homogeneous,linear differential equation and the initial conditions.E&U theorem implies that $y(x)=0$ is the only solution of the IVP.

b)(8 pts.) Can the function $W(x)=4(x-1)^2$ be the Wronskian on $(0,3)$ for $y''+p(x)y'+q(x)y=0$ (with p and q continuous)?Why?

$W(1)=0.\, W(y_1,y_2)(x)$ is either zero on (a,b) or different from zero $\forall x\in(a,b)$. Hence, $W(x)$ cannot be the Wronskian on $(0,3)$ for the equation.

c) (9 pts.) Write a transformation to transform

$$(x^2-2x+1)y'''+(5x-5)y''-3y'+\frac{2}{3(x-1)}y=e^x \quad , \quad x<1$$

into a 3rd order,linear ordinary differential equation with constant coefficients.Why?

After multiplying the equation by $(x-1)$ we obtain

$$(x-1)^3 y'''+5(x-1)y''-3(x-1)y'+\frac{2}{3}y=(x-1)e^x \quad , \quad x<1$$

which is a C-E equation.Hence $x-1=-e^t$ is the desired transformation.

2) (25 pts.) Find the largest interval (a,b) for which the E&U theorem guarantees the existence of a unique solution on (a,b) to the IVP

$$(x^3+8)y'''-2xy'+(\ln(1-x))y=\frac{e^x}{4-x} \quad , \quad y(-1)=3 \ , \ y'(-1)=-2 \ , \ y''(-1)=0.$$

$x^3+8,-2x,\ln(1-x), and \ \frac{e^x}{4-x}$ are continuous on $(-\infty,1)$, and x^3+8 vanishes at $x=-2$.

$x^3+8,-2x,\ln(1-x), and \ \frac{e^x}{4-x}$ are continuous on $(-2,1)$ which contains the initial point $x_0=-1$,and $a_0(x)=x^3+8$ is nowhere zero in this interval. So, $(a,b)=(-2,1)$.

3) a)(9 pts.) Solve $D(D-1)(D+2)y=0.$

The roots of the characteristic equation are $0,1, and -2$ which give $y(x)=c_1+c_2e^t+c_3e^{-2t}$.

b)(12 pts.) Using the method of undetermined of coefficients ,write an **appropriate form** of a particular solution y_p of

$$D(D-1)(D+2)y = 4e^t - \frac{1}{2} + 2\cos^2 t.$$

The complementary function y_c is $c_1 + c_2e^t + c_3e^{-2t}$.

The nonhomogeneous part $F(t) = 4e^t + \frac{1}{2} + \cos 2t$ is a solution of

$D(D-1)(D^2+4)y = 0.$

$\therefore y_p$ is in the form

$$A + Bt + (C + Et)e^t + Fe^{-2t} + G\cos 2t + H\sin 2t.$$

Since $1, e^t, and\ , e^{-2t}$ are linearly independent solutions of the homogeneous equation,we can delete them from y_p and take y_p as

$$Bt + Ete^t + G\cos 2t + H\sin 2t.$$

c) (4 pts.) Find only the coefficient of t appearing in the particular solution y_p in part (b) without evaluating the others.

$D(D-1)(D+2)(Bt) = \frac{1}{2} \Rightarrow B = -\frac{1}{4}.$

4) Consider $xy'' - (2x+1)y' + (x+1)y = F(x).$

a) (18 pts.) If e^x satisfies the corresponding homogeneous equation , find a fundamental set of solutions of the complementary equation.

Let $y_c(x) = e^x u(x)$, then $y_c'(x) = e^x u + e^x u'$,and $y_c''(x) = e^x u + 2e^x u' + e^x u''$.Substituting $y_c, y_c', and\ y_c''$ in the given equation we obtain

$$e^x(xu'' - u') = 0 \text{ or } xu'' - u' = 0.$$

Separating the variables,we obtain

$$\frac{dv}{v} = \frac{dx}{x} \text{ where } v' = u \text{ and } v \neq 0.$$

Integrating,we have

$$v = c_1 x \ \ , c_1 \in \mathbb{R}.$$

Then $u = c_2x^2 + c_3 \ , c_2, c_3 \in \mathbb{R}$

and $y_c(x) = (c_2x^2 + c_3)e^x$.

$y_1(x) = e^x$ and $y_2(x) = x^2e^x$ are linearly independent solutions of the complementary equation.

b) (7 pts.) We seek a particular solution of the nonhomogeneous equation of the form

$y_p = A(x)e^x + B(x)x^2e^x$ where $A(x)$ and $B(x)$ have to be determined.(i.e.,the method of V.P.)