1)(5 pts. each) Describe the following.

a)Order of a differential equation:The order of a differential equation is the order of the highest-order derivative appearing in the equation.

b)General form of a third-order linear equation:

$a_0(t)y^{(3)}+a_1(t)y''+a_2(t)y=f(t)$ *where* $a_0(t)$ *is not identically zero in some* interval I.

c)General form of an n-th order initial value problem:

$$F\left(t,y,y',y'',.....,y^{(n-1)},y^{(n)}\right)=0.............(A)$$

$$y(t_0)=c_0,y'(t_0)=c_1,........,y^{(n-1)}(t_0)=c_{n-1},c_i\text{'}s\in\mathbb{R}.$$

d) Euler's method for $y'=t^2-y^2$, $y(0)=0$:

$$y_n=y_{n-1}+h\left(t_{n-1}{}^2-y_{n-1}{}^2\right),n\ge 1.$$

e)Singular solution:

A solution of (A) which can not be obtained from any general soln.of (A) by any choice of the n essential arbitrary constants is called a singular soln. of (A).

2)Consider the IVP

$$y'=\frac{1}{(y+1)(t-2)} \quad , \quad y(0)=0.$$

a)(10pts.) Solve the IVP.

After separating variables of the given separable eqn., we obtain

$(y+1)dy=\frac{1}{t-2}dt$ and using direct integration

$(y+1)^2=2\ln(2-t)+c,c\in\mathbb{R}$ is obtained.

$y(0)=0\Rightarrow c=1-2\ln 2$

$$\Rightarrow y(t)=-1+\sqrt{1+2\ln\left(1-\frac{t}{2}\right)}.$$

b)(5 pts.) State the domain of definition of the solution.

$1+2\ln\left(1-\frac{t}{2}\right)\ge 0 \quad \Leftrightarrow t\le 2\left(1-e^{-\frac{1}{2}}\right)$.Thus,

$$D=\left(-\infty,2\left(1-e^{-\frac{1}{2}}\right)\right].$$

c)(5 pts.) Describe what happens to the solution as t approaches the limit of its domain of definition,i.e.,why can't the solution be extended for more time?

$y(t)\to -1$ where the eqn. doesn't exist as $t\to 2\left(1-e^{-\frac{1}{2}}\right)$ and

$y(t)\to\infty$ (doesn't exist) as $t\to -\infty$.

d)(5 pts.) Determine the interval in which the solution exist.

$$I=\left(-\infty,2\left(1-e^{-\frac{1}{2}}\right)\right)$$ which contains $t_0=0$.

3) Consider the differential equation

$$\left(\frac{y^2}{x}+\frac{y}{x^2}\right)dx+\left(y-\frac{1}{x}\right)dy=0.$$

a)(9 pts.) Find an integrating factor $\mu=\mu\left(\frac{y}{x}\right)$.

$\mu=\mu\left(\frac{y}{x}\right)$ is an integrating factor $\Leftrightarrow$ $\frac{\partial}{\partial y}(\mu M)=\frac{\partial}{\partial x}(\mu N)$ where $M(x,y)=\frac{y^2}{x}+\frac{y}{x^2}$ and $N(x,y)=y-\frac{1}{x}$

$$\Leftrightarrow \mu'(u)=\frac{-1}{u}\mu(u) \quad \text{where} \quad u=\frac{y}{x}$$

$$\Leftrightarrow \mu\left(\frac{y}{x}\right)=\frac{x}{y}$$ (i.e., any nonzero constant multiple of it can be taken as an integrating factor)

b)(8 pts.) Find the general solution of the equation.

After multiplying the eqn. by μ we obtain the exact eqn.

$$\left(y+\frac{1}{x}\right)dx+\left(x-\frac{1}{y}\right)dy=0.$$

Now,we have to find $\psi(x,y)$ s.t.

$$\psi_x(x,y)=y+\frac{1}{x}..............(1)$$

and

$$\psi_y(x,y)=x-\frac{1}{y}.....................(2)$$

From (1) $\psi(x,y)=xy+\ln|x|+h(y)$ and

$$\psi_y(x,y)=x+h'(y)................(3).$$

We obtain

$$\psi(x,y)=xy+\ln\left|\frac{x}{y}\right|+c, c\in\mathbb{R}$$ comparing $(2)\,and\,(3)$ and

the general soln. of the differential eqn. can be written as

$$xy+\ln\left|\frac{x}{y}\right|=A, A\in\mathbb{R}.$$

c)(8 pts.) Determine a solution y which satisfies $y(-1)=1.$

$y(-1)=1\Rightarrow A=-1$ and $\quad xy+\ln\left|\frac{x}{y}\right|+1=0.$

4)(25 pts.) Solve

$$y'-e^x y=y\ln y$$

by using the substitution $v=\ln y.$

The substitution v=lny transforms the eqn. into

$v'-v=e^x$ which is linear in v and $\mu(x)=e^{-x}$ can be taken as an integrating factor.

Thus $\frac{d}{dx}\left(e^{-x}v\right)=1$ and $v = xe^{x} + ce^{x}, c \in \mathbb{R}$ and

$\ln y = xe^{x} + ce^{x}$ is the general soln. of the eqn.