Math.240
HW IV
Due:December 13, 2005
Bring the homework to the lecture on the due date.

1) Solve $\left(1+x^2\right)y''+xy'-y=0$ around $x=0.$

The singular points are when $x=\pm i.$All other points are ordinary points.

$y(x)=\sum_{n=0}^{\infty}a_nx^n$, $|x|<1$ (Why?)

Substituting this assumed power series solution into the equation we obtain

$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=2}^{\infty}n(n-1)a_n+\sum_{n=1}^{\infty}na_n-\sum_{n=0}^{\infty}a_nx^n=0$, and

$a_2=\frac{1}{2}a_0$, $a_3=0$, and the recurrence equation $a_{n+2}=-\frac{n-1}{n+2}a_n$, $n\geq 2.$

Then, $a_{2k+1}=0,k\geq 1,$ and $a_{2k}=(-1)^k\frac{1.3.5.......(2k-3)}{2.4.6...........(2k)}a_0,k\geq 2.$(Using the recurrence equation).So

$$y(x)=a_0\left(1+\frac{1}{2}x^2+\sum_{n=2}^{\infty}(-1)^n\frac{1.3.5........(2n-3)}{2.4.6...........2n}\right)+a_1x.$$

2) Solve $y''-xy'+2y=2$ around the origin,and identify y_c and y_p.

$x=0$ is an ordinary point and the assumed solution is $y(x)=\sum_{n=0}^{\infty}a_nx^n$ .Substituting it into the equation we get $\sum_0^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_1^{\infty}na_nx^n+\sum_0^{\infty}2a_nx^n=2.$Then

$a_2=1-a_0$,and $a_{n+2}=\frac{(n-2)}{(n+1)(n+2)}a_n,n\geq 1.$Using the recurrence relation we obtain

$a_{2k}=0,k\geq 2$,and $a_{2k+1}=\frac{-1.3.5.......(2k-3)}{2.3.4......(2k+1)}a_1,k\geq 1.$Hence,

$$y(x)=a_0\left(1-x^2\right)+a_1(x+\sum_{n=1}^{\infty}\frac{-1.3.5.....(2n-3)}{2.3.4.....(2n+1)}x^{2n+1})+x^2, where$$

$y_p(x)=x^2$ and remaining part of $y(x)$ corresponds to y_c .

3) Find the first **four** nonzero terms in a power series expansion about $x_0=2$ for the general solution to $y''+(x-2)y'-y=0.$

We know that $y(x)=\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}\frac{y^{(n)}(2)}{n!}(x-2)^n$ .If $y(2)=c_1, and$ $y'(2)=c_2$,then $y''(2)=c_1$ can be obtained from the equation.Differentiating the equation

$y'''(2)=0, and \quad y^{(4)}(2)=y(2)$ is obtained. $\therefore \quad y(x)=c_1\left(1+\frac{(x-2)^2}{2}+\frac{(x-2)^4}{4!}\right)+c_2(x-2)$

is the required part of the solution.

4) How would you decide whether $x_0=1$ is a regular singular point or not for the **third-order**, linear, homogeneous differential equation

$$\left(x^3-3x+2\right)y''+\left[4\sin(x-1)\right]y''-y'+7(x+2)y=0.$$

Let's obtain the standard form of the equation

$$(x-1)^3 y'''+(x-2)^2\left[(x-1)\frac{4\sin(x-1)}{x^3-3x+2}\right]y''-(x-1)\left[(x-1)^2\frac{1}{x^3-3x+2}\right]y'+\left[(x-1)^3\frac{7(x+2)}{x^3-3x+2}\right]y=0.$$

Since

$\lim\limits_{x\to1}(x-1)\frac{4\sin(x-1)}{(x-1)^2(x+2)}=\frac{4}{3}$, $\lim\limits_{x\to1}(x-1)^2\frac{-1}{(x-1)^2(x+2)}=\frac{-1}{3}$,and

$\lim\limits_{x\to1}(x-1)^3\frac{7(x+2)}{(x-1)^2(x+2)}=0$ the point $x_0=1$ is a regular singular point.