Math.240
HW III
Due:November 25, 2005
Bring the homework to the lecture on the due date.

1 .Solve

$$2t^2y''+(y')^3=2ty' \quad , \quad t>0$$

using the substitution $v=y'$.

The given substitution transforms the diff. eqn. into

$v^{-3}v'-\frac{1}{v^2t}=-\frac{1}{2t^2}$ if $v\neq 0.$

Using the transformation $u=v^{-2}$, we obtain

$u'+\frac{2}{t}u=\frac{1}{t^2}$ which is linear in u.

$u=\frac{t+A}{t^2}, A\in\mathbb{R}$ is a soln.of the linear eqn.Using backsubstitutions

$v=\pm\frac{t}{\sqrt{t+A}}$ and $y=\pm\frac{2}{3}\sqrt{t+A}\,(t-A)+B, A\,\&\,B\in\mathbb{R}$, are obtained.

2 .Solve

$$y''+(y')^2=2e^{-y}$$

using a suitable transformation.(Hint:Play with the role of y.)

Method 1) The eqn. can be written as

$\frac{d^2}{dx^2}(e^y)=2$ which implies that

$e^y=(x+A)^2+B, A\,\&\,B\in\mathbb{R}.$

Method 2) If $y'=p(y)$,then $y''=p(y)\,p$ .Using this transformation,we obtain the following Bernoulli eqn.

$pp'(y)+p^2=2e^{-y}$, and its soln. is

$p^2=4e^{-y}+Ae^{-2y}$ .Then

$e^y y'=\sqrt{4e^y+A}$ (separable),and

$e^y=(x+B)^2+C$ can be found.

3 .Prove that

$\{e^{ax}, xe^{ax}, x^2e^{ax}, ..., x^{m-1}e^{ax}\}$ is a fundamental solution set of

$(D-a)^m y=0 \quad , \quad a\in\mathbb{R} \quad , \quad m\in\mathbb{Z}^+.$

Consider the linear relation

$e^{ax}\left(c_1+c_2x+.......+c_{n-1}x^{n-1}\right)=0 \quad ...... \quad (*)$.

Observation:

$$(D-a)e^{ax}f(x)=e^{ax}f'$$
$$(D-a)^2 e^{ax}f=e^{ax}f''$$
.
.
$$(D-a)^m e^{ax}f=e^{ax}f^{(m)}$$ where

$f(x)=c_1+c_2x+.....+c_m x^{m-1}$, and $f^m(x)=0$.

$\therefore y(x)=e^{ax}f(x)$ is a solution of $(D-a)^m y=0.$ .... (1)

The relation $(*)$ implies that

$$c_1+c_2x+.............+c_m x^{m-1}=0$$
$$c_2+2c_3x+...........+(m-1)c_m x^{m-2}=0$$
$$2c_3+........+(m-1)(m-2)c_m x^{m-3}=0$$
.
.
$$(m-1)(m-2)....(2)(1)c_m=0.$$

This system has only trivial solution (i.e., $c_1=c_2=....=c_m=0$) ....(2)

(1) & (2) $\Rightarrow$ $\{e^{ax}, xe^{ax}, x^2e^{ax}, ..., x^{m-1}e^{ax}\}$ is a fundamental set of solutions of $(D-a)^m y=0$

4 .Solve

$$(x-1)^3 y'''+(x-1)^2 y''-2(x-1)y'+2y=\ln|x-1|$$
$$y(0)=y'(0)=0, y''(0)=1.$$

If $1-x=e^t$,then $t=\ln(1-x)$ and the eqn. is transformed into

$$y'''(t)-2y''(t)-y'+2y=t.$$

Then, $y_c(t)=c_1e^t+c_2te^t+c_3e^{2t}$ and $y_p(t)=\frac{1}{4}+\frac{t}{2}.$

$\therefore y_g(x)=c_1(1-x)+c_2(1-x)\ln(1-x)+c_3(1-x)^2+\frac{1}{4}+\frac{\ln(1-x)}{2}$ is the general soln. of the C-E eqn.

5 .Find the **form** of the particular solution of $y_p(x)$ for

$$y^{(4)}+4y''=x\sin 2x+xe^x-1-8e^{-x}\cos 2x-5x.$$

(Don't evaluate the constants.)

The operator $L_1(D)=D^2(D^2+4)^2(D-1)^2((D+1)^2+4)$ annihilates the RS fcn.

Hence y_p is a soln. of

$$D^4\left(D^2+4\right)^3(D-1)^2\left((D+1)^2+4\right)y=0.$$

$\therefore y_p$ is of the form

$$Cx^2+Ex^3+\left(Gx+Hx^2\right)\cos 2x+\left(Jx+Kx^2\right)\sin 2x+\left(L+Mx\right)e^x+e^{-x}\left(N\cos 2x+P\sin 2x\right).$$