

## Math.240

### HW IV

Due: December 13, 2005

Bring the homework to the lecture on the due date.

1) Solve  $(1+x^2)y'' + xy' - y = 0$  around  $x = 0$ .

The singular points are when  $x = \pm i$ . All other points are ordinary points.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad |x| < 1 \quad (\text{Why?})$$

Substituting this assumed power series solution into the equation we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0, \quad \text{and}$$

$$a_2 = \frac{1}{2}a_0, \quad a_3 = 0, \quad \text{and the recurrence equation} \quad a_{n+2} = -\frac{n-1}{n+2}a_n, \quad n \geq 2.$$

$$\text{Then, } a_{2k+1} = 0, k \geq 1, \quad \text{and} \quad a_{2k} = (-1)^k \frac{1.3.5 \dots (2k-3)}{2.4.6 \dots (2k)} a_0, k \geq 2. \quad (\text{Using the}$$

recurrence equation). So

$$y(x) = a_0 \left( 1 + \frac{1}{2}x^2 + \sum_{n=2}^{\infty} (-1)^n \frac{1.3.5 \dots (2n-3)}{2.4.6 \dots 2n} \right) + a_1 x.$$

2) Solve  $y'' - xy' + 2y = 2$  around the origin, and identify  $y_c$  and  $y_p$ .

$x = 0$  is an ordinary point and the assumed solution is  $y(x) = \sum_{n=0}^{\infty} a_n x^n$ . Substituting it

into the equation we get  $\sum_0^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_1^{\infty} na_n x^n + \sum_0^{\infty} 2a_n x^n = 2$ . Then

$$a_2 = 1 - a_0, \quad \text{and} \quad a_{n+2} = \frac{(n-2)}{(n+1)(n+2)} a_n, \quad n \geq 1. \quad \text{Using the recurrence relation we obtain}$$

$$a_{2k} = 0, k \geq 2, \quad \text{and} \quad a_{2k+1} = \frac{-1.3.5 \dots (2k-3)}{2.3.4 \dots (2k+1)} a_1, k \geq 1. \quad \text{Hence,}$$

$$y(x) = a_0(1 - x^2) + a_1 \left( x + \sum_{n=1}^{\infty} \frac{-1.3.5 \dots (2n-3)}{2.3.4 \dots (2n+1)} x^{2n+1} \right) + x^2, \quad \text{where}$$

$$y_p(x) = x^2 \quad \text{and remaining part of } y(x) \text{ corresponds to } y_c.$$

3) Find the first **four** nonzero terms in a power series expansion about  $x_0 = 2$  for the general solution to  $y'' + (x-2)y' - y = 0$ .

We know that  $y(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{y^{(n)}(2)}{n!} (x-2)^n$ . If  $y(2) = c_1$ , and  $y'(2) = c_2$ , then

$y''(2) = c_1$  can be obtained from the equation. Differentiating the equation

$$y'''(2) = 0, \text{ and } y^{(4)}(2) = y(2) \text{ is obtained. } \therefore y(x) = c_1 \left( 1 + \frac{(x-2)^2}{2} + \frac{(x-2)^4}{4!} \right) + c_2(x-2)$$

is the required part of the solution.

**4)** How would you decide whether  $x_0 = 1$  is a regular singular point or not for the **third-order**, linear, homogeneous differential equation

$$(x^3 - 3x + 2)y'' + [4\sin(x-1)]y'' - y' + 7(x+2)y = 0.$$

Let's obtain the standard form of the equation

$$(x-1)^3 y''' + (x-2)^2 \left[ (x-1) \frac{4\sin(x-1)}{x^3 - 3x + 2} \right] y'' - (x-1) \left[ (x-1)^2 \frac{1}{x^3 - 3x + 2} \right] y' + \left[ (x-1)^3 \frac{7(x+2)}{x^3 - 3x + 2} \right] y = 0.$$

Since

$$\lim_{x \rightarrow 1} (x-1) \frac{4\sin(x-1)}{(x-1)^2(x+2)} = \frac{4}{3}, \quad \lim_{x \rightarrow 1} (x-1)^2 \frac{-1}{(x-1)^2(x+2)} = \frac{-1}{3}, \text{ and}$$

$$\lim_{x \rightarrow 1} (x-1)^3 \frac{7(x+2)}{(x-1)^2(x+2)} = 0 \text{ the point } x_0 = 1 \text{ is a regular singular point.}$$