

Math.240

HW I , Answers

1. Show that $\phi(x) = \sin x - \cos x$ is a solution to the IVP

$$y'' + y = 0; \quad y(0) = -1, \quad y'(0) = 1.$$

$$\phi'' + \phi = -\sin x + \cos x + \sin x - \cos x = 0, \text{ and } \phi(0) = -1 \text{ and } \phi'(0) = 1.$$

Therefore ϕ is a soln. of the IVP.

2.a) Determine whether the Existence and Uniqueness Theorem implies that the IVP

$$y' = (x-3)(y+1)^{\frac{2}{3}}; \quad y(1) = -1.$$

The IVP has a soln. since $f(x, y) = (x-3)(y+1)^{\frac{2}{3}}$ is cont. everywhere but the Existence and Uniqueness Thm. says nothing about uniqueness of its soln. (Why?)

b) Solve the IVP.

$$(y+1)^{-\frac{2}{3}} dy = (x-3) dx \quad \text{if } y \neq -1$$

$$\Rightarrow y = -1 + \left(\frac{x^2}{3} - 3x + c \right)^3, \quad c \in \mathbb{R}$$

$$y(1) = -1 \Rightarrow c = \frac{5}{2}$$

$$\therefore y(x) = -1 + \left(\frac{x^2}{2} - 3x + \frac{5}{2} \right)^3$$

On the other hand, $y = -1$ satisfies the IVP. Hence, $y = -1 + \left(\frac{x^2}{2} - 3x + \frac{5}{2} \right)^3$ and $y = -1$ are the solns.

3. Solve $(x+x^6)dy - ydx = 0$ by finding an integrating factor of the form

$$\mu(x, y) = x^\alpha y^\beta.$$

$$x^\alpha y^\beta (x+x^6)dy - x^\alpha y^{\beta+1}dx = 0 \quad (E)$$

$$\text{Then, } \frac{\partial}{\partial x} (y^\beta (x^{\alpha+1} + x^{\alpha+6})) = \frac{\partial}{\partial y} (-x^\alpha y^{\beta+1}) \Leftrightarrow \alpha = -6 \text{ and } \beta = 4.$$

Now, let's find $\psi(x, y)$ s.t.

$$\psi_x(x, y) = -x^{-6} y^5 \quad \dots \quad (A)$$

and

$$\psi_y(x, y) = x^{-6} y^4 (x+x^6) \quad \dots \quad (B)$$

$$(A) \Rightarrow \psi(x, y) = \frac{x^{-5} y^5}{5} + h(y)$$

$$(B) \Rightarrow h'(y) = y^4$$

$$\Rightarrow h(y) = y^5 + C, C \in \mathbb{R}$$

$$\therefore \psi(x, y) = \frac{x^{-5}y^5}{5} + \frac{y^5}{5} + C \text{ and } y^5(1+x^5) = Ex^5 \text{ is a general soln. of it.}$$

4. Find the continuous solution to the IVP

$$y' + p(x)y = x ; y(0) = 1,$$

where

$$p(x) = \begin{cases} 1 & \text{if } x \in [0, 2] \\ 3 & \text{if } x \in (2, \infty). \end{cases}$$

$$0 \leq x \leq 2 \Rightarrow y' + y = x, y(0) = 1$$

We know that any nonzero constant multiple of e^x can be taken as an integrating factor and the general soln. to the IVP can be written as

$$y(x) = x - 1 + 2e^{-x} \text{ if } x \in [0, 2]$$

$$x > 2 \Rightarrow y' + 3y = x$$

$$\text{and } \frac{d}{dx}(e^{3x}y) = xe^{3x}.$$

$$\text{This suggests that } y(x) = \frac{x}{3} - \frac{1}{9} + ce^{-3x} \text{ if } x > 2.$$

$$\lim_{x \rightarrow 2^-} y = \lim_{x \rightarrow 2^+} y = y(2) \text{ (for the continuity of the soln.)}$$

$$\Rightarrow c = e^6 \left(\frac{4}{9} + 2e^{-2} \right) \text{ and}$$

$$y(x) = \begin{cases} x - 1 + 2e^{-x} & \text{if } x \in [0, 2] \\ \frac{x}{3} - \frac{1}{9} + e^6 \left(\frac{4}{9} + 2e^{-2} \right) e^{-3x} & \text{if } x \in (2, \infty) \end{cases}$$

which is cont. but not differentiable at $x = 2$.

5. Use the substitution $y = vx^2$ to solve

$$y' = \frac{2y}{x} + x \cos \left(\frac{y}{x^2} \right).$$

Let $y = vx^2$ then $y' = 2xv + x^2v'$, and the eqn. is transformed into

$x^2v' = x \cos v$ which is separable. After separating variables and using integration, we obtain

$$\sec v + \tan v = Cx \text{ and using the back-substitution } y = vx^2$$

$$\sec \frac{y}{x^2} + \tan \frac{y}{x^2} = Cx, C \in \mathbb{R}.$$