1) Consider the third-order differential equation

$$(x^2-1)(x-2)y'''+(\sin x)y''-(2x+1)y'+3y=0.$$

a)(4 pts.) Determine the intervals where solutions exist.

All the coefficient functions are continuous $\forall x\in\mathbb{R}$,and $a_0(x)=(x^2-1)(x-2)$ vanishes at $x=\pm1,2.$ Then $I=(-\infty,-1)$,or $I=(-1,1)$,or $(1,2)$,or $I=(2,\infty).$

b)(4 pts.) Write down a related IVP.What does the E&U theorem tell you about this IVP?

$y(3)=y'(3)=0,y''(3)=8.$ In this case,the IVP has a unique solution on the interval $I=(2,\infty).$

c)(4 pts.) Let $y_1,y_2,and\ y_3$ be solutions of the above differential equation which satisfy $2y_1+3y_2-y_3=0.$ What is $W(y_1,y_2,y_3)$?Why?

$W(y_1,y_2,y_3)=0$ since $y_1,y_2,and\ y_3$ are linearly dependent.

d)(9 pts.) How did you decide whether $x_0=2$ is a reqular singular point or not.?

Since $\lim_{x\to2}\alpha(x)=\frac{\sin 2}{3},\lim_{x\to2}\beta(x)=0, and\ \lim_{x\to2}\gamma(x)=0, where$

$\alpha(x)=(x-2)\frac{\sin x}{(x^2-1)(x-2)}$ $,\beta(x)=(x-2)^2\frac{(2x+1)}{(x^2-1)(x-2)}, and$

$\gamma(x)=(x-2)^3\frac{3}{(x^2-1)(x-2)}$,exist, $x_0=2$ is a regular singular point of the equation.

2)a)(4 pts.) Write down an **eight-order,linear IVP** and **solve** it.

$y^{(8)}0=,y^{(n)}(1)=0,n=0,1,2,....,7$.

$y(x)=0.$

b)(2 pts.) Write down an example of a **nonlinear** ordinary differential equation of the **third-order**.

$y'''-yy'+8y=0.$

c)(4 pts.) What is the number of independent constants in the general solution of the linear system?

$$\begin{aligned}&y''+x'+x=\sin t\\&y'+x-y=0.\end{aligned}$$ Why?

order$\begin{vmatrix}D+1 & D^2\\ 1 & D-1\end{vmatrix}=$order$(-1)=0.$ So,the general solution of the system has no constants.

d)(7 pts.) Solve the linear system
$$\begin{aligned}&x'+\frac{3}{t}y'=0\\&x'-y''-\frac{1}{t^2}y=0,t>0.\end{aligned}$$

$x' = -\frac{3}{t}y'$ (from the first equation of the system).Substituting it into the second equation,we obtain $t^2y'' + 3ty' + y = 0, t > 0$ (C-E equation).Let $t = e^u$. Then $y''(u) + 2y'(u) + y(u) = 0$ is obtained.The general solution of the transformed equation is $y(u) = (A + Bu)e^{-u}$, $A \& B$ are constants. $y(t) = (A + B\ln t)\frac{1}{t}$ (using backsubstitution).And

$x(t) = \frac{3}{2}(B - A)t^{-2} + 3B\left(\frac{-\ln t}{2t} - \frac{1}{4t}\right) + C, A, B, C \in \mathbb{R}.$

3)(15 pts.) Solve the IVP $\left(\frac{2x}{y} - \frac{y}{x^2+y^2}\right)dx + \left(\frac{x}{x^2+y^2} - \frac{x^2}{y^2}\right)dy = 0, y(1) = 1.$

$\frac{\partial}{\partial y}\left(\frac{2x}{y} - \frac{y}{x^2+y^2}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2} - \frac{x^2}{y^2}\right) \Rightarrow$ exact.

Let's find $F(x, y)$ s.t.

$F_x = \frac{2x}{y} - \frac{y}{x^2+y^2} \Rightarrow F(x, y) = \frac{x^2}{y} - \arctan\frac{x}{y} + h(y), and$

$F_y = \frac{x}{x^2+y^2} - \frac{x^2}{y^2} = -\frac{x^2}{y^2} + \frac{x}{x^2+y^2} + h'(y).$

$\Rightarrow h(y) = A \in \mathbb{R}$ and $F(x, y) = \frac{x^2}{y} - \arctan\frac{x}{y} + A = B \in \mathbb{R}, or\ \frac{x^2}{y} - \arctan\frac{x}{y} = C \in \mathbb{R}.$

$y(1) = 1 \Rightarrow\ C = 1 - \frac{\pi}{4}$,and the solution of the IVP is $\frac{x^2}{y} - \arctan\frac{x}{y} = 1 - \frac{\pi}{4}.$

4)(20 pts.) Solve the IVP

$(1 + x^2)y'' - 4xy' + 6y = 0,\ \ y(0) = 1, y'(0) = -2.$

$x_0 = 0$ is an ordinary point.Then $y(x) = \sum_0^\infty a_n x^n, |x| < 1.$ Substituting the assumed power series solution into the equation,we obtain

$a_2 + 3a_0 = 0, 3a_3 + a_1 = 0, a_{n+2} = -\frac{(n-2)(n-3)}{(n+1)(n+2)}a_n, n \geq 2.$

$\Rightarrow a_2 = -3a_0, a_3 = -\frac{a_1}{3}, a_k = 0, k \geq 4.$

$\therefore y(x) = a_0(1 - 3x^2) + a_1\left(x - \frac{x^3}{3}\right).$

$y(0) = a_0 = 1, y'(0) = a_1 = -2 \Rightarrow y(x) = 1 - 3x^2 - 2\left(x - \frac{x^3}{3}\right).$

5)a)(3pts.) $\mathcal{L}\{ty'(t)\} = -\frac{d}{ds}(sY(s) - y(0)) = -sY'(s) - Y(s)$ where $Y(s) = \mathcal{L}\{y(t)\}.$

b)(4 pts.) $\mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \mathcal{L}\{1 * f(t)\} = \dfrac{F(s)}{s}, F(s) = \mathcal{L}\{f(t)\}.$

c)(7pts.) $\mathcal{L}^{-1}\left\{\dfrac{se^{-3s}}{s^2+4s+5}\right\} = \begin{cases} 0 & if \quad 0 \le t < 3 \\ e^{-2(t-3)}\left[\cos(t-3) - 2\sin(t-3)\right] & if \quad t \ge 3 \end{cases}$ where

$\mathcal{L}^{-1}\left\{\dfrac{s+2-2}{(s+2)^2+1}\right\} = e^{-2t}\left(\cos t - 2e^{-2t}\sin t\right).$

d)(2 pts.) $Y(s) = \mathcal{L}\{y(t)\}, \lim\limits_{s\to\infty} Y(s) = 0.$

e)(11 pts.) Solve the IVP $y'' + 3ty' - 6y = 1, y(0) = y'(0) = 0.$

Takig the Laplace transform of the equation,we obtain the following first-order linear ordinary differential equation

$Y'(s) + \dfrac{9-s^2}{3s}Y(s) = -\dfrac{1}{3s^2}$ with an integrating factor $\mu(s) = A\exp\int\left(\dfrac{3}{s} - \dfrac{s}{3}\right)ds = As^3e^{-\frac{s^2}{6}}$, A is a nonzero constant.Then,

$Y(s) = \dfrac{1}{s^3} + c\dfrac{s^{\frac{s^2}{6}}}{s^3}$.We know that

$$\lim_{s\to\infty} Y(s) = 0 \Leftrightarrow c = 0.$$
$$\Rightarrow Y(s) = \frac{1}{s^3}$$
$$\Rightarrow y(t) = \frac{t^2}{2}.$$