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1) Find the values of h, k, and $r > 0$ that make the circle $(x - h)^2 + (y - k)^2 = r^2$ tangent to the parabola $y = x^2 + 1$ at the point $(1, 2)$ and that also make the second derivatives d^2y/dx^2 have the same value on both curves there.

Parabola:

$y = x^2 + 1 \Rightarrow y' = 2x \Rightarrow y'' = 2$

$y'\big|_{(1,2)} = 2, \quad y''\big|_{(1,2)} = 2$

Circle:

We want: (1) $(1-h)^2 + (2-k)^2 = r^2$

(2) $2(x-h) + 2(y-k)\cdot\frac{dy}{dx} = 0 \xrightarrow{(x,y)=(1,2)} (1-h) + (2-k)\cdot 2 = 0$

(3) $1 + \left(\frac{dy}{dx}\right)^2 + (y-k)\cdot\frac{d^2y}{dx^2} = 0 \xrightarrow{(x,y)=(1,2)} 1 + 2^2 + (2-k)\cdot 2 = 0$

So: (3) $\Rightarrow \boxed{k = \frac{9}{2}}$

(2) $\Rightarrow \boxed{h = -4}$

(1) $\Rightarrow r^2 = \frac{125}{4} \Rightarrow \boxed{r = \frac{5\sqrt{5}}{2}}$

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2) A particle moves along the curve $2x^3+2y^3=9xy$. If the distance of the particle to the origin is increasing at a rate of 3 m/sec at the moment it passes through the point (2, 1), how fast is its x coordinate changing at this moment?

$$2x^3+2y^3=9xy$$

$$\Downarrow$$

$$6x^2\frac{dx}{dt}+6y^2\frac{dy}{dt}=9\frac{dx}{dt}y+9x\frac{dy}{dt}$$

$$\Downarrow (x,y)=(2,1)$$

$$8\frac{dx}{dt}+2\frac{dy}{dt}=3\frac{dx}{dt}+6\frac{dy}{dt} \Rightarrow 5\frac{dx}{dt}=4\frac{dy}{dt}$$

$$\Downarrow$$

$$\frac{dy}{dt}=\frac{5}{4}\frac{dx}{dt} \text{ at } (2,1).$$

$$L^2=x^2+y^2 \Rightarrow 2L\frac{dL}{dt}=2x\frac{dx}{dt}+2y\frac{dy}{dt}$$

$$\Downarrow (x,y)=(2,1)$$

$$\sqrt{5}\frac{dL}{dt}=2\frac{dx}{dt}+\frac{dy}{dt}=\frac{13}{4}\frac{dx}{dt}$$

$$\Rightarrow \frac{dx}{dt}=\frac{4\sqrt{5}}{13}\cdot\frac{dL}{dt}=\frac{4\sqrt{5}}{13}\cdot 3 \text{ m/sec}=\boxed{\frac{12\sqrt{5}}{13}\text{ m/sec}}$$

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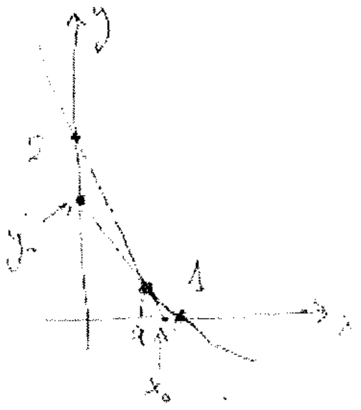
3) Let $P(a,b)$ be a point on the parabola $y = x^2 - 3x + 2$ such that $0 \le a \le 1$. Consider the tangent line to the parabola at P and let A denote the area of the triangle cut off from the first quadrant by this line. Find the maximum and the minimum possible values of A.



$y' = 2x - 3 \Rightarrow y'\big|_{x=a} = 2a - 3$

Tangent line:

$y - (a^2 - 3a + 2) = (2a - 3)(x - a)$

x-intercept:

$y = 0 \Rightarrow x_0 = \dfrac{a^2 - 2}{2a - 3}$

y-intercept: $x = 0 \Rightarrow y_0 = 2 - a^2$

$$A = \text{Area} = \frac{1}{2} x_0 y_0 = -\frac{1}{2} \cdot \frac{(a^2 - 2)^2}{2a - 3} \quad \text{for} \quad 0 \le a \le 1$$

$$\frac{dA}{da} = -\frac{1}{2} \cdot \frac{2(a^2 - 2) \cdot 2a(2a - 3) - (a^2 - 2)^2 \cdot 2}{(2a - 3)^2} = -\frac{(a^2 - 2)(3a^2 - 6a + 2)}{(2a - 3)^2} = 0$$

$\Rightarrow a = \pm\sqrt{2},\ 1 \pm \frac{1}{\sqrt{3}} \Rightarrow$ Only $\underbrace{a = 1 - \frac{1}{\sqrt{3}}}_{\text{critical point}}$ is in $(0, 1)$

$\boxed{A = \dfrac{4}{3\sqrt{3}}}$ ← maximum

Endpoints:

$a = 0 \Rightarrow A = \dfrac{2}{3}$

$a = 1 \Rightarrow A = \dfrac{1}{2}$ ← minimum

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4a) Evaluate the limit $\lim_{x \to a} \frac{\sin x - \sin a}{\sin(x-a)}$.

$$\lim_{x \to a} \frac{\sin x - \sin a}{\sin(x-a)} = \lim_{x \to a} \left(\frac{\sin x - \sin a}{x-a} \cdot \frac{x-a}{\sin(x-a)} \right)$$

$$= \lim_{x \to a} \frac{\sin x - \sin a}{x-a} \cdot \lim_{x \to a} \frac{x-a}{\sin(x-a)}$$

$$= \left. \frac{d}{dx}(\sin x) \right|_{x=a} \cdot 1 = \cos a$$

4b) Find y' if $y = \tan^5(x^3)$. (Do not simplify!)

$$y' = 5 \tan^4(x^3) \cdot \sec^2(x^3) \cdot 3x^2$$

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5a) Assume that f is continuous on $[0,2]$ and $f(0) = f(2) = 0$. Show that $f(c+1) = f(c)$ for some c.

Consider $g(x) = f(x+1) - f(x)$ on $[0,1]$. g is continuous.

$g(0) = f(1) - f(0) = f(1)$

$g(1) = f(2) - f(1) = -f(1)$

If $f(1) = 0$, then $f(1) = f(0)$ and we take $c = 0$.

If $f(1) \neq 0$, then g has opposite signs at 0 and 1,

therefore, by IVT, $g(c) = 0$ for some c in $(0,1)$.

This means $f(c+1) = f(c)$.

5b) Assume that g is differentiable on $(0,\infty)$ and $\lim_{x\to\infty} g'(x) = 11$. Evaluate $\lim_{x\to\infty} \frac{g(5x) - g(2x)}{x}$.

Apply MVT to g on $[a,b] = [2x, 5x]$:

There is a c in $(2x, 5x)$ such that

$$\frac{g(5x) - g(2x)}{5x - 2x} = g'(c)$$

Hence:

$$\lim_{x\to\infty} \frac{g(5x) - g(2x)}{x} = 3 \lim_{x\to\infty} g'(c) = 3 \cdot \lim_{c\to\infty} g'(c) = 3 \cdot 11 = 33$$

because $c > 2x$

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