MTI	Q1	Q2	Q3	Q4	Q5

1) For a real number t, let L_t be the line passing through the point $(-1,0)$ with slope t, and let M_t be the line passing through the point $(1,0)$ and perpendicular to L_t.

a) Find the intersection point P_t of the lines L_t and M_t.

If $t \neq 0$, then:

$$\left.\begin{array}{l} L_t : \ y = t(x-1) \\ M_t : \ y = -\frac{1}{t}(x+1) \end{array}\right\} \Rightarrow t(x-1) = -\frac{1}{t}(x+1)$$

$$\Downarrow$$

$$x = \frac{1-t^2}{1+t^2}$$

$$y = \frac{2t}{1+t^2}$$

$$\Downarrow$$

If $t=0$, then:

$\underbrace{L_0 : y = 0 \Rightarrow M_0 : x = 1}$

$$\Downarrow$$

$P_0 = (1,0)$

$$\boxed{P_t = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right) \text{ for all } t}$$

b) Describe the curve traced by P_t for $-\infty < t < \infty$ using Cartesian coordinates.

$$\left(\frac{1-t^2}{1+t^2}\right)^2 + \left(\frac{2t}{1+t^2}\right)^2 = \frac{1-2t^2+t^4+4t^2}{(1+t^2)^2} = \frac{1+2t^2+t^4}{(1+t^2)^2} = 1$$

$$\Rightarrow \quad x^2+y^2 = 1.$$

$x = \frac{1-t^2}{1+t^2} = 1 - \frac{2}{1+t^2}$ takes on all values in $(-1,1]$

and for these x, $y = \frac{2t}{1+t^2}$ gives all possible values of y in $x^2+y^2=1$.

Hence $\boxed{x^2+y^2=1, \ (x,y) \neq (-1,0)}$ is the trajectory.

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2) Determine all values of the constant a for which the function $f(x) = e^x + ae^{-x}$ is one-to-one. For these values of a, find $f^{-1}(x)$ and the domain of f^{-1}.

We solve $y = e^x + ae^{-x}$ for x:

$$(e^x)^2 - y \cdot e^x + a = 0 \Rightarrow e^x = \frac{y \pm \sqrt{y^2-4a}}{2}$$

$$\Rightarrow x = \ln\left(\frac{y \pm \sqrt{y^2-4a}}{2}\right)$$

Case 1: $a > 0$.
If $y > \sqrt{4a}$, then $y + \sqrt{y^2-4a} > y - \sqrt{y^2-4a} > 0$, and there are two solutions. So f is not 1-1 if $a > 0$.

Case 2: $a = 0$.
$y = e^x \Rightarrow x = \ln y$ is the unique solution for $y > 0$.
Hence f is 1-1 and $f^{-1}(x) = \ln x$ for $x > 0$ if $a = 0$.

Case 3: $a < 0$.
For all y, $y + \sqrt{y^2-4a} > 0 > y - \sqrt{y^2-4a}$. So there is a unique x for any y.
Hence f is 1-1 and $f^{-1}(x) = \ln\left(\frac{x + \sqrt{x^2-4a}}{2}\right)$ for all x if $a < 0$.

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3) Find all real solutions of the equation $27^x - 3^{2x+1} + 3^{x+1} = 1$.

$$\Updownarrow$$

$$27^x - 3^{2x+1} + 3^{x+1} - 1 = 0$$

$$\Updownarrow$$

$$3^{3x} - 3 \cdot 3^{2x} + 3 \cdot 3^x - 1 = 0$$

$$\Updownarrow$$

$$(3^x)^3 - 3(3^x)^2 + 3 \cdot 3^x - 1 = 0$$

$$\Updownarrow$$

$$(3^x - 1)^3 = 0$$

$$\Updownarrow$$

$$3^x - 1 = 0$$

$$\Updownarrow$$

$$x = 0$$

$x = 0$ is the only solution.

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4) Express $\sin(5\theta)$ in terms of $\sin\theta$.

$$\sin 5\theta = \sin(\theta + 4\theta)$$

$$= \sin\theta \cdot \cos 4\theta + \cos\theta \cdot \sin 4\theta$$

$$= \sin\theta \cdot (1 - 2\sin^2 2\theta) + \cos\theta \cdot (2\sin 2\theta \cos 2\theta)$$

$$= \sin\theta \cdot (1 - 2 \cdot (2\sin\theta\cos\theta)^2) + \cos\theta \cdot 2 \cdot (2\sin\theta\cos\theta) \cdot (1 - 2\sin^2\theta)$$

$$= \sin\theta \cdot (1 - 8\sin^2\theta\cos^2\theta) + 4\sin\theta\cos^2\theta \cdot (1 - 2\sin^2\theta)$$

$$= \sin\theta \cdot (1 - 8\sin^2\theta \cdot (1 - \sin^2\theta)) + 4\sin\theta \cdot (1 - \sin^2\theta) \cdot (1 - 2\sin^2\theta)$$

$$= \boxed{5\sin\theta - 20\sin^3\theta + 16\sin^5\theta}$$

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5) Find the following limits:

a) $\lim_{x \to -1} \frac{\sqrt{\cancel{x^3}^{\,x^2} + 8} - 3}{x + 1} = \lim_{x \to -1} \frac{\sqrt{x^2+8} - 3}{x+1}$

$(x^2+8) - 9 = x^2 - 1 = \cancel{(x+1)}(x-1)$

$$= \lim_{x \to -1} \frac{\overbrace{(\sqrt{x^2+8} - 3)(\sqrt{x^2+8} + 3)}}{\cancel{(x+1)}(\sqrt{x^2+8} + 3)}$$

$$= \lim_{x \to -1} \frac{x-1}{\sqrt{x^2+8} + 3}$$

$$= \frac{-1-1}{\sqrt{(-1)^2+8} + 3} = \boxed{-\frac{1}{3}}$$

b) $\lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} = \lim_{h \to 0} \frac{x^2 - (x+h)^2}{h \cdot (x+h)^2 \cdot x^2}$

$$= \lim_{h \to 0} \frac{-2xh + h^2}{h \cdot (x+h)^2 \cdot x^2}$$

$$= \lim_{h \to 0} \frac{-2x + h}{(x+h)^2 \cdot x^2}$$

$$= \frac{-2x + 0}{(x+0)^2 \cdot x^2} = \boxed{-\frac{2}{x^3}}$$

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