1)a) $\lim_{x\to 1}\dfrac{\int_1^x e^{\sin(\frac{\pi t}{2})}dt}{\ln x} \to \dfrac{0}{0}$ using L'HR and FTC,

$$= \lim_{x\to 1}\frac{e^{\sin(\frac{\pi x}{2})}}{\frac{1}{x}} = e.$$

b) $\lim_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{1/x^2} \to 1^\infty$

$= \lim_{x\to 0} e^{\frac{1}{x^2}\ln(\frac{\sin x}{x})} = e^{\lim_{x\to 0}\frac{\ln(\frac{\sin x}{x})}{x^2}} = e^L$ where

$L = \lim_{x\to 0}\dfrac{\ln(\frac{\sin x}{x})}{x^2} \to \dfrac{0}{0}$ using L'HR

$= \lim_{x\to 0}\dfrac{\frac{x}{\sin x}(\frac{x\cos x-\sin x}{x^2})}{2x} = \lim_{x\to 0}\dfrac{x\cos x-\sin x}{2x^2\sin x} \to \dfrac{0}{0}$ using L'HR

$= \lim_{x\to 0}\dfrac{\cos x - x\sin x - \cos x}{4x\sin x + 2x^2\cos x} = \lim_{x\to 0}\dfrac{-x\sin x}{x(4\sin x + 2x\cos x)} = \lim_{x\to 0}\dfrac{-\sin x}{4\sin x + 2x\cos x} \to \dfrac{0}{0}$

using L'HR

$= -\lim_{x\to 0}\dfrac{\cos x}{4\cos x + 2\cos x - 2x\sin x} = -\dfrac{1}{6}$ and $e^L = e^{-1/6}$.

2) a) Evaluate $\int \frac{dx}{(x^2+2x+2)^{3/2}}$.

$$\int \frac{dx}{(x^2+2x+2)^{3/2}} = \int \frac{dx}{\left((x+1)^2+1\right)^{3/2}} \qquad \begin{aligned} t &= x+1 \\ dt &= dx \end{aligned}$$

$$= \int \frac{dt}{(t^2+1)^{3/2}} \qquad \begin{aligned} t &= \tan\theta \\ dt &= \sec^2\theta d\theta \end{aligned} \qquad \text{where } \frac{-\pi}{2} < \theta < \frac{\pi}{2}$$

$$= \int \frac{\sec^2\theta d\theta}{(\sec^2\theta)^{3/2}} = \int \cos\theta d\theta = \sin\theta + C$$

$$= \frac{t}{\sqrt{t^2+1}} + C = \frac{x+1}{\sqrt{(x+1)^2+1}} + C$$

b) Evaluate $\int \frac{x}{\cos(x^2)} dx$ using the substitution $\begin{aligned} u &= x^2 \\ du &= 2xdx \end{aligned}$

$$= \frac{1}{2}\int \frac{du}{\cos u} = \frac{1}{2}\int \sec u du = \frac{1}{2}\ln\left|\sec u + \tan u\right| + C = \frac{1}{2}\ln\left|\sec(x^2) + \tan(x^2)\right| + C$$

3)a) Find a positive number a satisfying $\int_0^a \frac{dx}{1+x^2} = \int_a^\infty \frac{dx}{1+x^2}$

$$\int_0^a \frac{dx}{1+x^2} = \arctan x\Big|_0^a = \arctan a \qquad \text{(I)}$$

$$\int_a^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty}\int_a^b \frac{dx}{1+x^2} = \lim_{b\to\infty}\arctan b - \arctan a = \frac{\pi}{2} - \arctan a \qquad \text{(II)}$$

Equate (I) and (II):

$$\arctan a = \frac{\pi}{2} - \arctan a \Rightarrow \frac{\pi}{4} = \arctan a \Rightarrow a = 1.$$

b) Find the Maclaurin series of $f(x) = \frac{5x}{2x^2+x-3}$.

$$\frac{5x}{2x^2+x-3} = \frac{A}{x-1} + \frac{B}{2x+3} \Rightarrow A = 1, B = 3.$$

$$\frac{5x}{2x^2+x-3} = \frac{1}{x-1} + \frac{3}{2x+3} = \frac{-1}{1-x} + \frac{3}{2}\frac{1}{x+\frac{3}{2}} = \frac{-1}{1-x} + \frac{3}{2}\frac{2}{3}\frac{1}{1-(-\frac{2}{3}x)}$$

$$\frac{-1}{1-x} = -\sum_{n=0}^{\infty} x^n \text{ where } |x|<1 \text{ and}$$

$$\frac{1}{1-(-\frac{2}{3}x)} = \sum_{n=0}^{\infty}(-1)^n(\frac{2}{3}x)^n \text{ where } |x| < \frac{3}{2}$$

$$\frac{5x}{2x^2+x-3} = \sum_{n=0}^{\infty}[(-1)^n(\frac{2}{3})^n - 1]x^n \quad \text{where} \quad |x|<1.$$

4) Consider the power series $\sum_{n=0}^{\infty} \frac{(n+1)(2x+1)^n}{(2n+1)2^n}$.

Find the radius of convergence and interval of convergence of this power series. Determine whether the power series is **absolutely convergent**, **conditionally convergent** or **divergent** at the **left end-point** and at the **right end-point** of its interval of convergence.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+2)(2x+1)^{n+1}}{(2n+3)2^{n+1}}\frac{2^n(2n+1)}{(n+1)(2x+1)^n}\right| = \left|\frac{2x+1}{2}\right| < 1 \Rightarrow |2x+1| < 2 \Rightarrow \frac{-3}{2} < x < \frac{1}{2}$$

Thus the radius of convergence R = 1

Check the end points:

$x = \frac{-3}{2} \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)}{(2n+1)}$ and $\lim_{n\to\infty}\frac{n+1}{2n+1} = \frac{1}{2} \neq 0.$ Thus the series diverges by the n-th term test.

$x = \frac{1}{2} \Rightarrow \sum_{n=0}^{\infty} \frac{(n+1)}{(2n+1)}$ and it is again divergent by the n-th term test.

Interval of convergence I = $(\frac{-3}{2}, \frac{1}{2})$.

5)a) Find an equation of the line L of intersection of the pair of planes $4x + y + z = 0$ and $2x + 3y - 2z = -5$.

$\vec{n}_1 x \vec{n}_2 = \begin{vmatrix} i & j & k \\ 4 & 1 & 1 \\ 2 & 3 & -2 \end{vmatrix} = -5i + 10j + 10k$ and it is parallel to $\vec{v} = i - 2j - 2k$ is the direction vector of the line .To find a point on the line we give $z = 0$ the solving two equations $\begin{matrix} 4x + y = 0 \\ 2x + 3y = -5 \end{matrix}$ we get $y = -2$ and $x = \frac{1}{2}$. Thus one of the intersection points is ($\frac{1}{2}, -2, 0)$.

The parametric equation of the line will be

$(x, y, z) = (\frac{1}{2} + t, -2 - 2t, -2t)$ where $-\infty < t < \infty$.

You may write the symmetric equation also. It will be

$$\frac{x - \frac{1}{2}}{1} = \frac{y + 2}{-2} = \frac{z}{-2}.$$

b) Write the equation of the plane through $P(1,1,1)$ which is perpendicular to the line L in part **a)**.

Normal vector of the plane is parallel to the direction vector of the line Then

(x-1, y-1, z-1) • (1,-2,-2) = 0

x - 2y - 2z +3 = 0 is the equation of the plane.