Date: 30 April 2005
Time: 14:00-15:45

MATH 112
MIDTERM EXAM II

SURNAME	
NAME	
SECTION	
STUDENT NO	
SIGN	

IMPORTANT:

1) This exam consists of 5 questions of **equal** weights.
2) Please read the questions carefully and write your answers under the corresponding question. Be neat.
3) **Show all your work**. Correct answers without sufficient explanation might **not** get full credit.
4) Calculators and dictionaries are not allowed.
5) Close your cellular telephones.

1	2	3	4	5	TOTAL
20	**20**	**20**	**20**	**20**	**100**

1) Determine whether each of the following series is convergent or divergent. State clearly the name and the conditions of the test you are using.

a) $\sum_{n=3}^{\infty} \frac{2n^2+3}{(n+1)(n-2)}$

Since $\lim_{n\to\infty} \frac{2n^2+3}{(n+1)(n-2)} = 2 \neq 0$ then by the n-th term test the given series diverges.

b) $\sum_{n=1}^{\infty} \frac{n^2(n)!}{(2n)!}$

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2(n+1)!}{(2n+2)!} \frac{(2n)!}{n^2(n)!} = \lim_{n\to\infty} \frac{(n+1)^2}{2n^2(2n+1)} = 0 < 1$

then by the Ratio Test the given series converges.

2) Determine whether each of the following series is convergent or divergent. State clearly the name and the conditions of the test you are using.

a) $\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)^n$

Since $\lim_{n\to\infty}\sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n}=\frac{2}{3}<1$ then by the Root Test the given series converges.

b) $\sum_{n=1}^{\infty}\sin^2\left(\frac{1}{n}\right)$

1st Solution:

$\lim_{n\to\infty}\frac{\sin^2\left(\frac{1}{n}\right)}{\frac{1}{n^2}}=1>0$ (i.e. it is $\neq 0$ and $\neq\infty$) and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by the p-test, p=2>1, then the given series converges by the Limit Comparison Test.

2nd Solution:

Since $\left|\sin^2\left(\frac{1}{n}\right)\right|<\frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by the p-test, p=2>1

then $\sum_{n=1}^{\infty}\sin^2\left(\frac{1}{n}\right)$ converges <u>absolutely</u> and $\sum_{n=1}^{\infty}\sin^2\left(\frac{1}{n}\right)$ converges.

3)a) Either evaluate the improper integral $\int_{2}^{\infty} \frac{x^2+17x-8}{(2x+3)(x-1)^2}\, dx$ or show that it is divergent.

Since the improper integral $\int_{2}^{\infty} \frac{1}{x}\, dx$ diverges by the p-test , p=1, and

$\lim_{x\to\infty} \frac{\frac{x^2+17x-8}{(2x+3)(x-1)^2}}{\frac{1}{x}} = \frac{1}{2} > 0$ then by the Limit Comparison Test for Improper Integrals the given integral diverges.

b)Evaluate the limit $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^4}$ by **using series.**

$$\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^4} = \frac{(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots)-(1-\frac{x^2}{2}+\frac{x^4}{4(2!)}-(\frac{x^2}{2})^3\frac{1}{3!}+\cdots)}{x^4} = -\frac{1}{12}$$

4) Consider the power series $\sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{2n}}{(n+2)4^n}$.

Find the radius of convergence and interval of convergence of this power series. Determine whether the power series is **absolutely convergent**, **conditionally convergent** or **divergent** at the **left end-point** and at the **right end-point** of its interval of convergence.

$$\lim_{n\to\infty} \left| \frac{(x-1)^{2n+2}}{(n+3)4^{n+1}} \frac{(n+2)4^n}{(x-1)^{2n}} \right| = \frac{(x-1)^2}{4}$$

and the series absolutely converges if $\frac{(x-1)^2}{4} < 1 \Rightarrow |x-1| < 2 \Rightarrow -1 < x < 3$

The series is absolutely convergent when $x \in (-1,3)$ and radius of convergence is R=2.

End Points:

If $x = 3$ then we get the alternating series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+2)}$. This series satisfies the conditions of A.S.T.

1) $u_n = \frac{1}{n+2} > 0$ when $n \geq 0$

2) $\frac{1}{(n+3)} < \frac{1}{(n+2)} \Rightarrow u_{n+1} < u_n$.

3) $\lim_{n\to\infty} \frac{1}{n+2} = 0$.

Thus the series converges by A.S.T.

On the other hand $\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{n+2} \right| = \sum_{n=0}^{\infty} \frac{1}{n+2}$ is a divergent harmonic series. Thus the alternating series is conditionally convergent.

If $x = -1$ then we get the same alternating series as in the previous case.
Therefore I=[-1,3] is the interval of convergence.

5)a) Find the sum of the series $\sum_{n=0}^{\infty} \frac{1}{(n+1)5^n}$ exactly.

1st Solution:

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ where $|x| < 1$. Integrate both sides with respect to x:

$\int \frac{1}{1-x} dx = \int \left(\sum_{n=0}^{\infty} x^n \right) dx$ where $|x| < 1 \Rightarrow -\ln(1-x) + C = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)}$ where $|x| < 1$.

Putting $x=0$ we can find that $C = 0$.

Thus $-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)}$ where $|x| < 1$.

Dividing both sides by x,

$\frac{-\ln(1-x)}{x} = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)}$ where $|x| < 1$ and $x \neq 0$.

Put $x = \frac{1}{5}$;

$$\frac{-\ln(1-\frac{1}{5})}{\frac{1}{5}} = -5\ln(\frac{4}{5}) = 5\ln(\frac{5}{4}) = \sum_{n=0}^{\infty} \frac{1}{(n+1)5^n}$$

2nd Solution:

You may also use the power series of $\frac{1}{1+x} = \sum (-1)^n x^n$ where $|x| < 1$. In this case you should put $x = \frac{-1}{5}$ at the end.

b) If $\sum_{n=1}^{\infty} a_n$ is convergent and $a_n > 0$ for all n then show that $\sum_{n=1}^{\infty} \frac{2a_n}{3+a_n}$ converges.

If $\sum a_n$ is a convergent then by the n-th Term Test $\lim_{n \leftarrow \infty} a_n = 0$. Using Limit Comparison Test we find that

$\lim_{n \to \infty} \frac{\frac{2a_n}{3+a_n}}{a_n} = \frac{2}{3} > 0$. Since $\sum a_n$ is a convergent then the given series is also convergent by the Limit Comparison Test.