

Senior Research Project Final Report: Analysis of Two-Temperature Steady State Phase Transitions

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A two-dimensional non-equilibrium Ising Model with individual spins and their evaluation over time is analyzed in this work. The spins are assumed to be interacting with two different heat baths. The spin states are changing randomly, with transition rates that depend on their nearest and next nearest neighboring spins. This specific kind of system is defined by Glauber dynamics. Since there are two heat reservoirs in this case, the system displays non-equilibrium steady-state behavior. The process is also defined as a Markoff process with their corresponding transition rates and spin state probabilities. Furthermore, we analyze the phase transitions in the system between the ferromagnetic and antiferromagnetic states of the lattice using a mean-field approximation approach and also a renormalization group solution.

1. Introduction

The statistical mechanics involving highly interacting particles have been the subject of many studies. The effect of changing the macroscopic properties of the system is a long-standing discussion. The fluctuations in the temperature in a many-particle system with interacting particles can indicate the properties of the system while displaying the important phenomena of phase transitions. In this work, we assume a closed system with two heat baths that reaches a non-equilibrium steady state. Therefore, there are two competing dynamics involved in the system due to the two different heat reservoirs. So, our main goal is to investigate the behavior of such a system while focusing on a 2-dimensional Ising model.

We will implement Glauber Dynamics[1] as well as a Markoff process with their corresponding transition rates. It has been shown[2] that this model is of the same universality as the equilibrium Ising class. Therefore, one can follow a similar model for this system. In this work, we will take into consideration the nearest-neighbor interactions between the particles as

well as their next-nearest-neighbor interactions. Also, we will implement the Mean Field Approximation in order to reduce the model to a simpler one. The main idea is to observe the criticality of the system when there are fluctuations in the temperatures by means of the corresponding phase transitions and the magnetic properties of the system.

Additionally, we also implement a solution using a position-space Renormalization Group (PSRG) transformation. The renormalization group transformation allows us to scale the lattice to get to a so-called "course-grained" system [5]. In this procedure, groups of spins are combined and treated as effective degrees of freedom at a larger length scale. The idea is that this iterative transformation is universal, so that it does not change the behaviour of the system as it extends to infinity. In this work, we also utilize a decimation method known as the Migdal-Kadanoff approximation, where we move the bonds inside the scaled unit, outside of it. With this method, we analyze the properties of the system by means of a map of the coupling constants consisting of several trivial and critical fixed points found by a numerical algorithm. Furthermore, using the critical fixed points, that indicates a phase transition of the system, we classify different phases of the lattice and also calculate the universal value for the correlation length exponent.

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2. The Mean Field Approximation

Let us start with our solution with the Mean Field approximation.

2.1. The Model

To introduce the system, let us define the properties of the model. We consider a 2-dimensional Ising lattice. The possible spin states are given as $S_i = \pm 1$. The Hamiltonian of the system can be expressed as the following.

$$H = -J\beta \sum_{(i,j)} S_i S_j$$

where β is the inverse temperature, J is the coupling constant, and this is valid for the nearest-neighbor interactions only. Now, say that $P_i(t)$ is the probability of the state with spin S_i at time t . The probability functions evolve corresponding to the Markoff process governed by the following master equation[4].

$$\frac{d}{dt} P_i(t) = \sum_{j \neq i} [P_j(t) \omega(j \rightarrow i) - P_i(t) \omega(i \rightarrow j)]$$

and,

$$\frac{d}{dt} \mathbf{P}(t) = \mathcal{L} \mathbf{P}(t)$$

where $\mathbf{P}(t)$ is the column matrix consisting of the state probabilities and \mathcal{L} is the Liouville matrix of the system satisfying the master equation. And the condition

$$\sum_{i=1}^N P_i(t) = 1$$

holds for all possible N states of the system. And the expectation of the corresponding spin values are given as,

$$\langle S_i \rangle = m_i$$

where m_i is the magnetization of the i^{th} spin and it satisfies the condition,

$$m_i(t) = \sum_{S_i = \pm 1} P_i(t) S_i$$

In our case, we are working with particles that can take the spin values $S_i = \pm 1$. Then for just one particle case, the equation becomes,

$$m(t) = P_1(t) \cdot (+1) + P_2(t) \cdot (-1)$$

The possible transition rates for a Glauber process are given as[3],

$$\omega = \alpha [1 - \tanh(J\beta S_i \sum_{|j-i|=1} S_j)]$$

2.2. Single Spin System

Our implementation of the model starts with a single spin in a 2-dimensional lattice (see Figure 1). For this simple case, we only consider the nearest-neighbor interactions and use the mean-field approximation to assume that all four nearest sites evolve with magnetization $m(t)$. Then, for the energies of this system, we can write the following.

$$H = -K_{nn} \sum_{(i,j)} S_i S_j$$

where we replaced the interaction constant with $K_{nn} = \frac{J}{k_B T} = J\beta$. Then, assuming the expected values of the spins of the nearest neighbors to be simply m , we get the following energy difference between a spin-up and a spin-down particle.

$$\Delta E = 4mK_{nn} - (-4mK_{nn}) = 8mK_{nn}$$

Plugging this into the Glauber transition rates, we get,

$$\omega_{1 \rightarrow 2} = \alpha [1 - \tanh(4K_{nn}m)]$$

$$\omega_{2 \rightarrow 1} = \alpha [1 + \tanh(4K_{nn}m)]$$

where state 1 corresponds to spin-up ($S_i = +1$) and state 2 corresponds to spin-down ($S_i = -1$) states of the system. And where $\omega_{1 \rightarrow 2}$ and $\omega_{2 \rightarrow 1}$ are the probability of flipping spin from state 1 to 2 and from state 2 to 1 per unit time, consecutively.

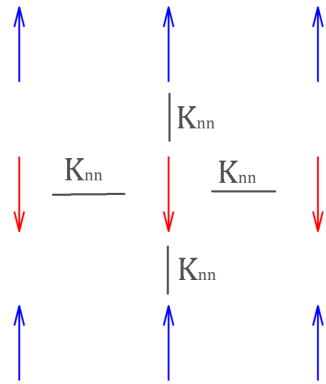


FIG. 1: Nearest neighbor interactions on a single spin.

The master equation for the system is given as,

$$\frac{d}{dt} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} -\omega_{1 \rightarrow 2} & \omega_{2 \rightarrow 1} \\ \omega_{1 \rightarrow 2} & -\omega_{2 \rightarrow 1} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

where P_1 is the probability of finding the particle in state 1 and P_2 is the probability of finding the particle in state 2. As well as the magnetization of the particle being satisfied by,

$$m(t) = P_1(t) \cdot (+1) + P_2(t) \cdot (-1) = P_1(t) - P_2(t)$$

In order to solve this set of first-order differential equations, we implement a numerical solution. First, solving for the relaxation of the system over time for a specific set of coupling constants, we then move on to the steady-state magnetization of the single spin for different values the coupling constant.

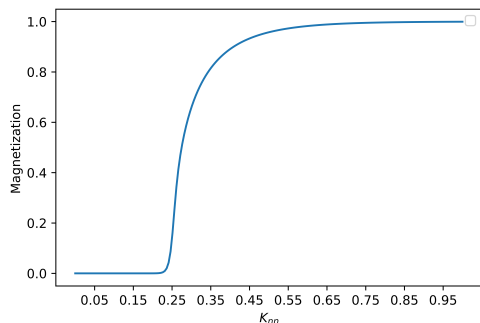


FIG. 2: The magnetization of the single particle plotted versus K_{nn} values. The critical behavior is observed around $K_{nn} = 0.25$. The initial conditions given to the system are as follows: $\alpha = 2$, and $P_1(t = 0) = 0.6$, $P_2(t = 0) = 0.4$, and $m(t = 0) = 0.2$.

From the plot in Figure 2, we observe that the magnetization value in the steady state is 0 until K_{nn} reaches the critical threshold of 0.25. After that threshold value, we observe that the magnetization starts to go to 1. This criticality indicates the phase transition of the system. The system transitions from a paramagnetic order to a ferromagnetic order. Before K_{nn} reaches the critical threshold, the magnetization value is zero or near zero, which indicates that the probabilities of the two states are nearly equal and therefore there is disorder in the lattice, corresponding to a paramagnetic state. After the critical threshold, one can see that the magnetization of the spin is equal to 1, which happens specifically when the probability of the spin-up state, P_1 , goes to 1 and the probability of the spin-down state, P_2 , goes to 0; indicating a ferromagnetic distribution. So, the lattice is filled only with spin-up particles in this order, while the expectation value of the spin of the particle equals to a value of +1. It is also important to note that K_{nn} and the temperature of the system are inversely proportional. So the reverse of the relation between the magnetization and K_{nn} is valid for the relation in between magnetization and temperature. The higher the temperature, the lower the interaction constant K_{nn} , and vice versa. From the results of our work in this section, we observe that as K_{nn} gets higher values, the nearest neighbor interactions in the lattice are more dominant, and hence the ferromagnetic properties of the system are reinforced and all the spins are pointing upwards.

2.3. Two Spin System

In this section, we worked on the same model mentioned before. But we have extended our work to a system of two spins located inside a 2-dimensional lattice again. We start by defining the spins of the two particles as S_1 and S_2 , as well as their magnetization values $m_1 = \langle S_1 \rangle$ and $m_2 = \langle S_2 \rangle$ respectively.

We work on two different lattices for this case, one is an antiferromagnetic one and; the other is neither a ferromagnetic one nor an antiferromagnetic one, let us refer to this phase of the system as "wavy". For the first case, let us work on the wavy lattice.

2.3.i. The Wavy Lattice

In this section, we define a lattice that can be seen in Figure 3, and call it the "wavy" lattice.

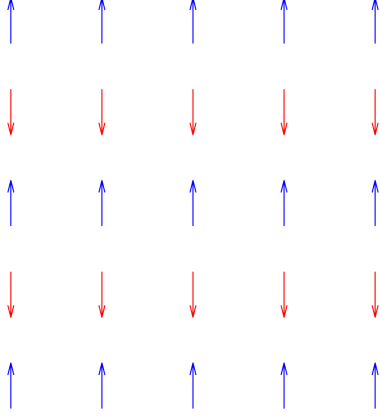


FIG. 3: The 2- dimensional lattice in the "wavy" phase.

For this system, we consider the nearest-neighbor interactions as well as the next-nearest-neighbor interactions in the lattice (see Figure 4).

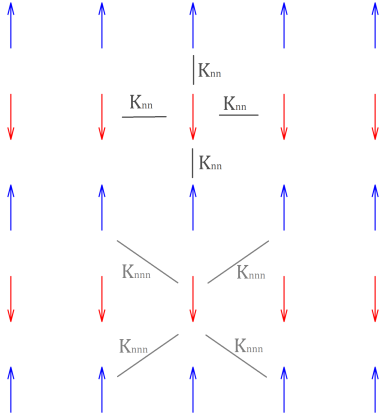


FIG. 4: The nearest neighbor (K_{nn}) and the next nearest neighbor interactions (K_{nnn}) involved in the lattice.

Now, let us choose two arbitrary spins inside the lattice and name them S_1 and S_2 , and give the neighboring atoms their corresponding magnetization values m_1 and m_2 , as in Figure 5.

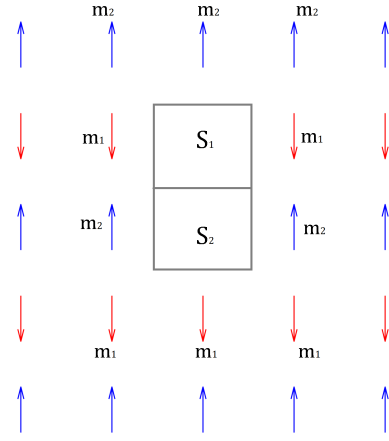


FIG. 5: Magnetization values distributed over the nearest neighbors and the next nearest neighbors of the two spins S_1 and S_2 . Here the down spins are assigned a magnetization value of m_1 and the up spins are assigned a magnetization value of m_2 .

Since there are now two spins that we consider, the number of possible states of such configuration is 4. The 4 possible states can be seen in Figure 6, as well as the transition rates that drive the system. At each transition, we consider that only one spin flips (with the rates $\omega_{1 \rightarrow 2}$, $\omega_{1 \rightarrow 3}$, $\omega_{3 \rightarrow 4}$, and $\omega_{2 \rightarrow 4}$, as well as their reverses) at a time. One other dynamic of the system we consider is one of the spins hopping from the first one to the second and vice versa ($\omega_{2 \rightarrow 3}$ and its reverse $\omega_{3 \rightarrow 2}$). So, if the state of the two-spin system is down-up, it becomes up-down, or if it is up-down initially, then it becomes down-up.

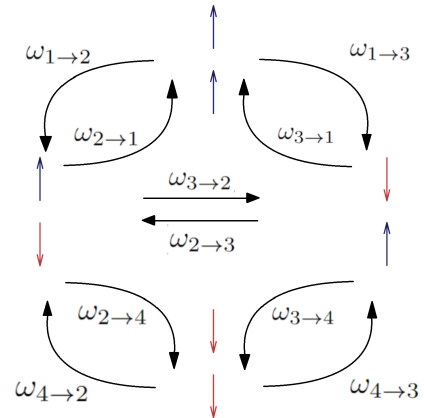


FIG. 6: The four possible states of the system and the corresponding transition rates between the states, where the first state is up-up, the second is up-down, the third is down-up, and the fourth is down-down.

The Markovian process of the system can be expressed as the following,

$$\frac{d}{dt} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} = \begin{bmatrix} -\omega_{1 \rightarrow 2} - \omega_{1 \rightarrow 3} & \omega_{2 \rightarrow 1} & \omega_{3 \rightarrow 1} & 0 \\ \omega_{1 \rightarrow 2} & -\omega_{2 \rightarrow 1} - \omega_{2 \rightarrow 3} - \omega_{2 \rightarrow 4} & \omega_{3 \rightarrow 2} & \omega_{4 \rightarrow 2} \\ \omega_{1 \rightarrow 3} & \omega_{2 \rightarrow 3} & -\omega_{3 \rightarrow 1} - \omega_{3 \rightarrow 2} - \omega_{3 \rightarrow 4} & -\omega_{4 \rightarrow 3} \\ 0 & \omega_{2 \rightarrow 4} & \omega_{3 \rightarrow 4} & -\omega_{4 \rightarrow 2} - \omega_{4 \rightarrow 3} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} \quad (1)$$

By calculating the corresponding energies of each state using the configuration in Figure 5, we derive the expressions for each of the transition rates. For the sake of simplicity, we express the inverse transition of each rate with the plus minus sign, since the only difference between the two rates is the sign the of the tanh argument.

$$\omega_{\frac{1 \rightarrow 2}{2 \rightarrow 1}} = \alpha [1 \mp \tanh[K_{nn}(m_1 + 2m_2 + 1) + 4K_{nnn}m_1]]$$

$$\omega_{\frac{1 \rightarrow 3}{3 \rightarrow 1}} = \alpha [1 \mp \tanh[K_{nn}(2m_1 + m_2 + 1) + 4K_{nnn}m_2]]$$

$$\omega_{\frac{2 \rightarrow 4}{4 \rightarrow 2}} = \alpha [1 \mp \tanh[K_{nn}(2m_1 + m_2 - 1) + 4K_{nnn}m_2]]$$

$$\omega_{\frac{3 \rightarrow 4}{4 \rightarrow 3}} = \alpha [1 \mp \tanh[K_{nn}(m_1 + 2m_2 - 1) + 4K_{nnn}m_1]]$$

$$\omega_{\frac{2 \rightarrow 3}{3 \rightarrow 2}} = \alpha [1 \pm \tanh[\delta(K_{nn} - 4K_{nnn})(m_1 - m_2)]]$$

The relations for the magnetization values are again derived by the expectation value of each spin, and they are given as the following.

$$m_1 = \langle S_1 \rangle = P_1 + P_2 - P_3 - P_4 \quad (2)$$

$$m_2 = \langle S_2 \rangle = P_1 - P_2 + P_3 - P_4 \quad (3)$$

Where we have the δ multiplier inside the last two transition rates. The multiplier corresponds to the ratio between the two temperatures (T_1/T_2) so that it cancels out with the T_1 inside the argument of the tanh. Then, we define the rates $\omega_{\frac{2 \rightarrow 3}{3 \rightarrow 2}}$ to be affected by the second heat bath only and all the other rates to be governed by the first heat bath.

We have solved this system of differential equations by implementing a numerical solution by plugging in the expressions for the transition rates. We have plotted the steady-state magnetization values for both m_1 and m_2 with different values for K_{nn} and K_{nnn} in order to observe the criticality of the system. The phase transitions of the system are evidently present, and can be observed by the color changes in each phase diagram.

The two plots in Figure 7 visualize the two different values for magnetization separately, where we have used the identities $K_{nn} = K$ and $K_{nnn} = G$. Combining the two, we can see that in the region on the left hand-side $m_1 = 1$ and $m_2 = -1$, and therefore it is a "wavy" phase of the lattice. In the middle region, we see that $m_1 = m_2 = 0$ and thus it is a paramagnetic phase. And lastly in the region on the right hand side, we see that $m_1 = m_2 = 1$ which corresponds to a ferromagnetic phase in the lattice. Classifying these regions, we observe that it makes sense to combine these two plots in the same graph for the sake of simplicity. And therefore, in further figures we use the product of m_1 and m_2 , where $m_1 \cdot m_2 = -1$ indicates a "wavy" phase, and $m_1 \cdot m_2 = 0$ indicates a paramagnetic phase, and $m_1 \cdot m_2 = 1$ indicates a ferromagnetic phase.

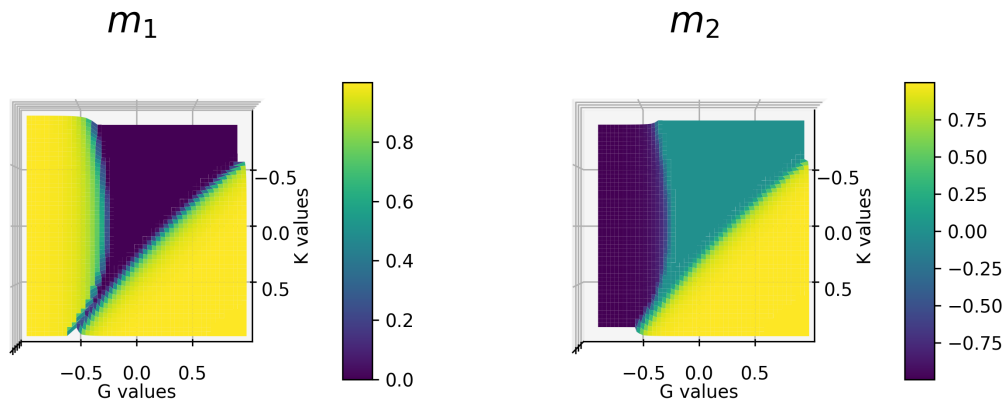
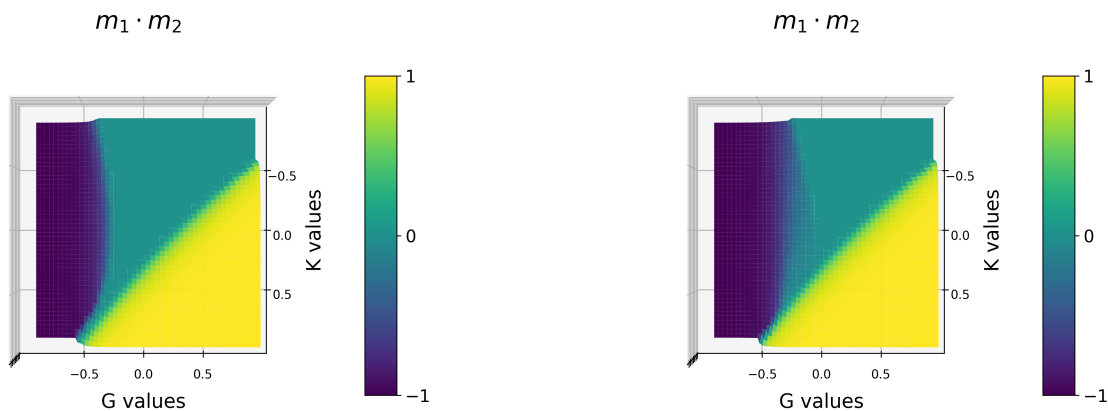


FIG. 7: The steady-state values for m_1 and m_2 plotted separately for $\alpha = 2$, and initial conditions $P_1(t = 0) = 0.3$, $P_2(t = 0) = 0.3$, $P_3(t = 0) = 0.2$, and $P_4(t = 0) = 0.2$, and equal temperatures for the two heat baths such that: $T_1 = T_2$, for the wavy lattice. The plots are 3-D but the captured from the birds-eye-view, thus a color-bar is provided.



(a) The steady-state values for m_1 and m_2 plotted together in the same graph as a product of the two values for the given parameters and equal temperatures for the two heat baths such that: $T_1 = T_2$.

(b) The steady-state values for m_1 and m_2 plotted together in the same graph as a product of the two values for the given parameters and non-equal temperatures for the two heat baths such that: $T_1 = 10T_2$.

FIG. 8: The steady-state magnetization values plotted for $\alpha = 2$, and initial conditions $P_1(t = 0) = 0.3$, $P_2(t = 0) = 0.3$, $P_3(t = 0) = 0.2$, and $P_4(t = 0) = 0.2$ for the wavy lattice. The plots are 3-D but they are captured from the birds-eye-view, thus a color-bar is provided.

Commenting on the changes, as the second temperature decreases, we can see that the phase transition line between the paramagnetic and "wavy" phases both fade and shift a little to the right, while the phase transition line between the paramagnetic and ferromagnetic phases does not change. This shift to the right means that the lattice going from a wavy order to a paramagnetic one, now requires a higher value for the constant K_{nnn} and thus, stronger interactions between the next nearest neighbors. Overall for the $T_1 = 10T_2$ case, we observe a larger area given to the wavy phase of the system. And although the contrary ($T_1 < T_2$) is not shown here

as a figure, in fact the opposite happens and the same line shifts a little to the left, leaving more area for the paramagnetic phase of the lattice.

Additionally, the physical meaning of attributing negative values to the interaction constants can be explained by a heat bath at a negative temperature, which can be evaluated as an engine that pumps energy into the system. This energy flux is what creates the orders that differentiate this specific system from being an Ising model that reaches equilibrium, but rather a steady-state.

2.3.ii. The Anti-ferromagnetic Lattice

Now, working on the antiferromagnetic (AF) lattice, we have the configuration in Figure 9.

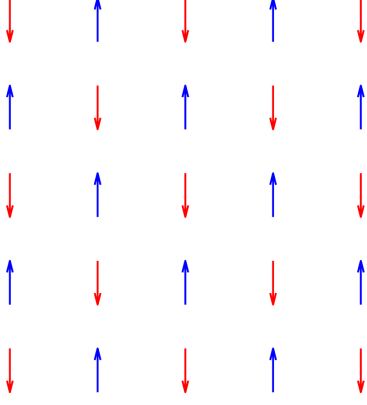


FIG. 9: The 2-dimensional lattice in the AF phase.

And we still have the nearest neighbor interactions and the next nearest neighbor interactions in the lattice, as before (see Figure 4). Again, working on two arbitrary spins S_1 and S_2 , we have the following configuration in Figure 10 with the magnetization of the spins that surround them.

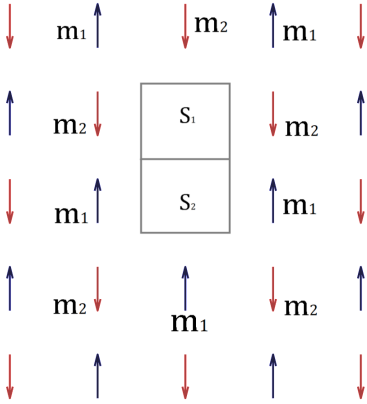


FIG. 10: Magnetization values distributed over the nearest neighbors and the next nearest neighbors of the two spins S_1 and S_2 . Here the up spins are assigned a magnetization value of m_1 and the down spins are assigned a magnetization value of m_2 .

And the four possible states and their corresponding transitions are identical to the one in Figure 6, only with different values for the specific transition rates. The same Markovian process also holds here, and so the Master Equation 1 holds as well. The transition rates for this

lattice type is given as the following and they are calculated from the energy of each state.

$$\omega_{\frac{1 \rightarrow 2}{2 \rightarrow 1}} = \alpha[1 \mp \tanh[K_{nn}(3m_1 + 1) + 4K_{nnn}m_2]]$$

$$\omega_{\frac{1 \rightarrow 3}{3 \rightarrow 1}} = \alpha[1 \mp \tanh[K_{nn}(3m_2 + 1) + 4K_{nnn}m_1]]$$

$$\omega_{\frac{2 \rightarrow 4}{4 \rightarrow 2}} = \alpha[1 \mp \tanh[K_{nn}(3m_2 - 1) + 4K_{nnn}m_1]]$$

$$\omega_{\frac{3 \rightarrow 4}{4 \rightarrow 3}} = \alpha[1 \mp \tanh[K_{nn}(3m_1 - 1) + 4K_{nnn}m_2]]$$

$$\omega_{\frac{2 \rightarrow 3}{3 \rightarrow 2}} = \alpha[1 \pm \tanh[\delta(3K_{nn} - 4K_{nnn})(m_2 - m_1)]]$$

And then using the same solution as before, we get the phase transitions for the AF lattice as in Figure 11, separately for m_1 and m_2 . But this time, we display the plots not from the top but from a different perspective to address the dimensionality of the plots. From this perspective, one can clearly see the critical surface associated with the coupling constants of the system.

In Figure 12, the products of the two magnetization values can be seen, where $m_1 \cdot m_2 = -1$ indicates an AF phase, and $m_1 \cdot m_2 = 0$ indicates a paramagnetic phase, and $m_1 \cdot m_2 = 1$ indicates a ferromagnetic phase. Thus, one can see that the blue region in Figure 12a, is a paramagnetic phase, whereas the purple and yellow regions correspond to antiferromagnetic and ferromagnetic phases respectively. Also, it is observed that as the second temperature (T_2) decreases, both transition lines move higher in the plots. For the phase shift from the ferromagnetic order to the antiferromagnetic order, keeping the constant K_{nnn} the same, it now requires higher values for the K_{nn} coupling constant. Thus, stronger interactions between the nearest neighbors are required now, in order to transition from a ferromagnetic lattice to an AF lattice, which is an interesting result. Furthermore, it also requires stronger interactions between both the nearest and next nearest neighbors to transition from a paramagnetic lattice to an antiferromagnetic one, leaving less area for the AF phase in the overall scheme.

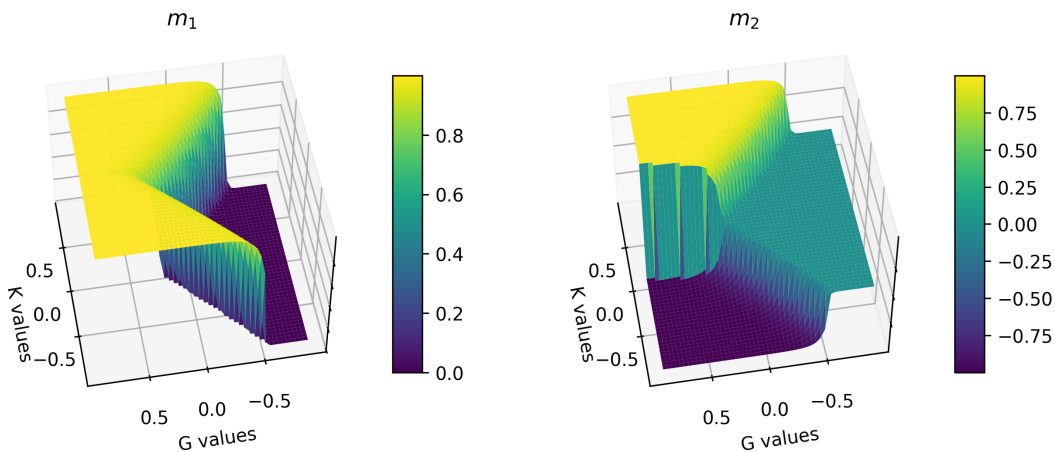
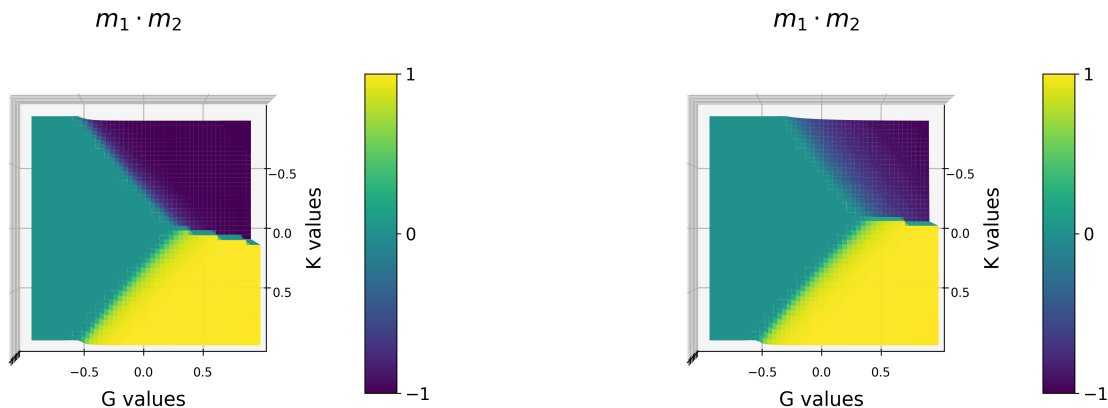


FIG. 11: The steady-state values for m_1 and m_2 plotted separately for $\alpha = 2$, and initial conditions $P_1(t = 0) = 0.3$, $P_2(t = 0) = 0.3$, $P_3(t = 0) = 0.2$, and $P_4(t = 0) = 0.2$, and equal temperatures for the two heat baths such that: $T_1 = T_2$, for the AF lattice. The plots are 3-D but the captured from the birds-eye-view, thus a color-bar is provided.



(a) The steady-state values for m_1 and m_2 plotted together in the same graph as a product of the two values for the given parameters and equal temperatures for the two heat baths such that: $T_1 = T_2$.

(b) The steady-state values for m_1 and m_2 plotted together in the same graph as a product of the two values for the given parameters and non-equal temperatures for the two heat baths such that: $T_1 = 5T_2$.

FIG. 12: The steady-state magnetization values plotted for $\alpha = 2$, and initial conditions $P_1(t = 0) = 0.3$, $P_2(t = 0) = 0.3$, $P_3(t = 0) = 0.2$, and $P_4(t = 0) = 0.2$, for the AF lattice. The plots are 3-D but the captured from the birds-eye-view, thus a color-bar is provided.

Our results from the Mean Field Theory clearly show the phase transitions of both lattices and how they change within the two temperature case. In both of our results, the critical surfaces are found. However, with this technique we were not able to get a result that is able to scan both phases (wavy and AF) within the same plot. Assigning values to the magnetization of the particles as m_1 and m_2 , goes only so far and one can only see the wavy, ferromagnetic and paramagnetic phases while modeling the wavy lattice as a starting point and vice versa for the antiferromagnetic lattice. This limitation

to the solution with the Mean Field approximation is inevitable for the 2-spin case. However, analyzing more spins at once (i.e. 4-spins) and therefore constructing more states, would allow the solutions to be more comprehensive. But still, one should note that as the size of the problem gets bigger, the computation time is also longer for further calculations.

3. The RG Transformation

One other solution that we developed is the Renormalization Group transformation. Since with the Mean Field approximation we were not able to construct critical surfaces that were valid for both the lattices that we worked on, with this solution we try to achieve that. That is to say, we try to form a RG transformation that holds for both lattice types (wavy and AF), and find the relevant critical lines from there.

3.1. The Model

Using the same Hamiltonian as before, we have,

$$H = -J\beta \sum_{(i,j)} S_i S_j$$

and the PSRG transformation function is defined such that,

$$\mathcal{R}(\mathbf{K}) \rightarrow \mathbf{K}'$$

where starting with a two-dimensional point \mathbf{K} , it takes to point \mathbf{K}' . And for a fixed point \mathbf{K}^* we have the following.

$$\mathcal{R}(\mathbf{K}^*) \rightarrow \mathbf{K}^*$$

And the correlation length is defined as $\xi = \frac{1}{2}e^{J/k_B T}$ [6]. Assuming that the correlation length diverges with some exponent (ν), then the correlation length varies with $\xi(T) \sim (T - T_c)^{-\nu}$ as $T \rightarrow T_c$, where T_c is the critical temperature.

Switching from \mathbf{K} to \mathbf{K}' , it should hold that the corresponding partition functions are still equivalent, such that $\mathbf{Z} = \mathbf{Z}'$. We have the following initial partial function.

$$\mathbf{Z} = \sum_{S_i = \pm 1} e^{K_{nn} \sum_{\langle n,n. \rangle} S_i S_j + K_{nnn} \sum_{\langle n,n.n. \rangle} S_i S_j}$$

and the scaled partition function,

$$\mathbf{Z}' = \sum_{\sigma_i = \pm 1} e^{K'_{nn} \sum_{\langle n,n. \rangle} \sigma_i \sigma_j + K'_{nnn} \sum_{\langle n,n.n. \rangle} \sigma_i \sigma_j}$$

where K_{nn} is the coupling constant for the nearest neighbor interactions and K_{nnn} is the coupling constant for the next nearest neighbor interactions and the primed versions are the scaled constants. Also, starting from a lattice with spins $S_i = \pm 1$, we scale it to $\sigma_i = \pm 1$ with a scaling constant of $b = \sqrt{5}$.

Demonstrating our transformation in both lattice types in Figures 13 and 14, one can see that this transformation is indeed valid for both, since the scaled lattice is equivalent to the initial one in each case.

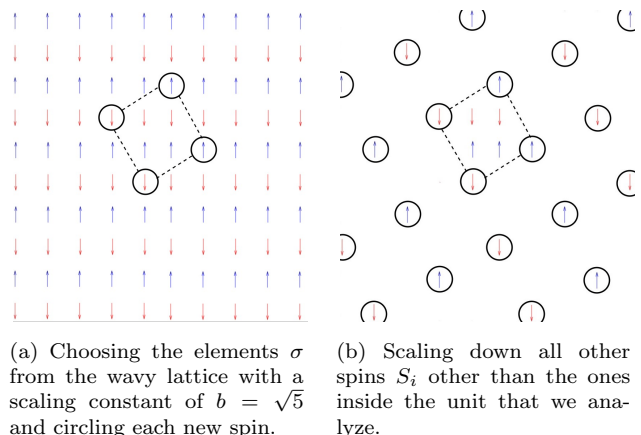


FIG. 13: The renormalization group transformation of the wavy lattice where each initial particle has the spin S_i , and in the transform each particle inside the circle has the spin σ_1 . One can see that in the aftermath the lattice with the circles is equivalent to a wavy lattice again.

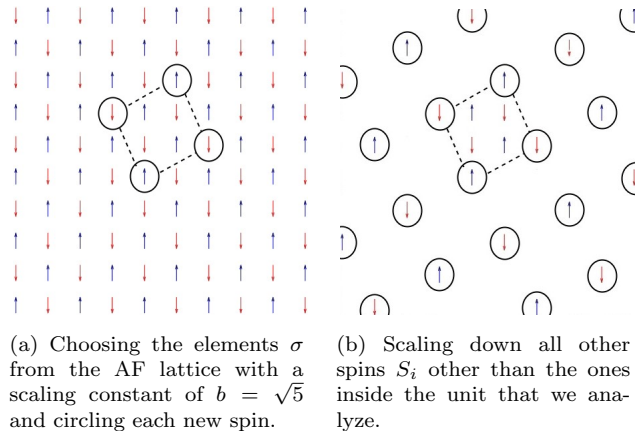


FIG. 14: The renormalization group transformation of the AF lattice where each initial particle has the spin S_i , and in the transform each particle inside the circle has the spin σ_1 . One can see that in the aftermath the lattice with the circles is equivalent to an AF wavy lattice again.

Now, zooming in one of the cells that we scale, we

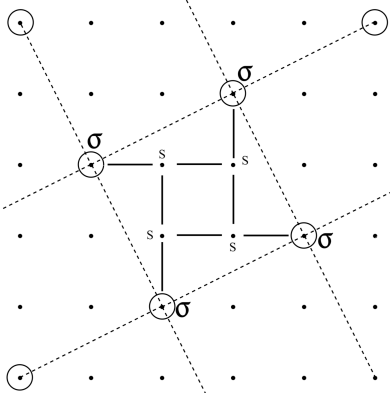


FIG. 15: Naming each spin in the scaled lattice as σ , and showing the nearest neighbor interactions inside the cell.

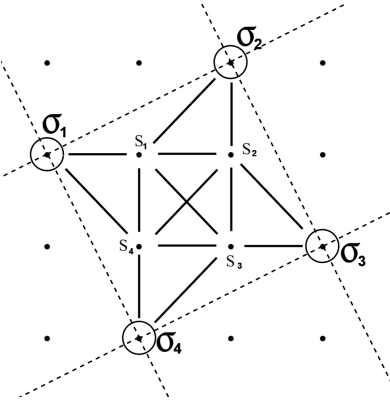


FIG. 16: Applying the Migdal-Kadanoff bond moving approximation to the lattice. Here the black lines show the nearest and next nearest neighbor interactions inside one cell of the scaled lattice.

get to the Figure 15 where we have shown the nearest neighbor interactions in the initial lattice with black lines and the nearest neighbor interactions in the scaled lattice with dashed lines.

From there, we name the spins from 1 to 4 in order to construct the partition functions of the initial and the scaled lattices. We implement the Migdal-Kadanoff bond moving approximation and carry each bond inside the cell shown in Figure 16 outside of the cell. Here we utilize the ratio of the total interaction number (with the interactions that cross the borders of the cell) to the ones (that are completely) inside the cell in order to scale the coupling constants. Overall, this procedure is called "decimation".

After the decimation process, we start to construct the partition functions of each lattice. The partition function of the initial lattice is given as,

$$\mathbf{Z} = \sum_{S_i=\pm 1} e^{K_{nn}(\sigma_1 S_1 + S_1 S_2 + S_4 S_3 + \sigma_3 S_3 + S_1 S_4 + S_2 S_3 + \sigma_2 S_2)} \cdot e^{K_{nnn}(\sigma_2 S_1 + S_2 S_4 + \sigma_4 S_3 + \sigma_1 S_4 + S_1 S_3 + \sigma_3 S_2)} \quad (4)$$

which should be equal to the following.

$$\mathbf{Z}' = \sum_{\sigma_i=\pm 1} e^{K'_{nn}(\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_4 + \sigma_1 \sigma_4) + K'_{nnn}(\sigma_1 \sigma_3 + \sigma_2 \sigma_4)} \quad (5)$$

Using equations 4 and 5, we implement a numerical solution to this system. The fact that performing one RG transformation takes us to another one, gives a new set of coupling constants at each iteration. And after trying a few initial points in the region where $K_{nn} > 0$ and $K_{nnn} > 0$, a critical fixed point is found and can be seen in Figure 17

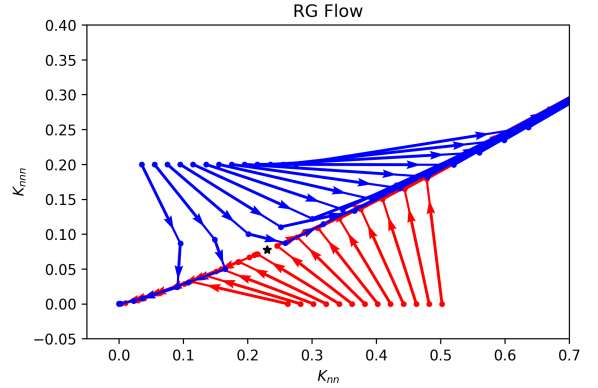


FIG. 17: The RG flow of several points in the region $K_{nn}, K_{nnn} > 0$. The critical fixed point is marked as a black star.

In Figure 17, we observe 2 fixed points and one critical fixed point. The black star indicates the critical fixed point and it is observed around $K_{nn} = 0.230$ and $K_{nnn} = 0.077$. The critical fixed point is defined by its almost repulsive nature in the RG flow graphs. If the chosen initial point is not directly on the critical line, then critical point is never reached by the consecutive transformations, instead the the points get closer and closer and then repel from it. If the initial point is chosen on the left-hand side of the critical line then they go to the fixed point at the origin, which is the high temperature fixed point indicating the paramagnetic phase of the lattice where there is disorder. If the initial point

is chosen on the right-hand side of the critical line, then they go the fixed point at both coupling constants going to infinity, which is a low temperature fixed point with a ferromagnetic phase in the lattice due to the strong coupling interactions ($K_{nn} \gg 0$ and $K_{nnn} \gg 0$).

Additionally, the critical exponent ν is also calculated. The universal value for the 2-dimensional Ising model is given as 1, and we have found the result: $\nu \approx 0.74$.

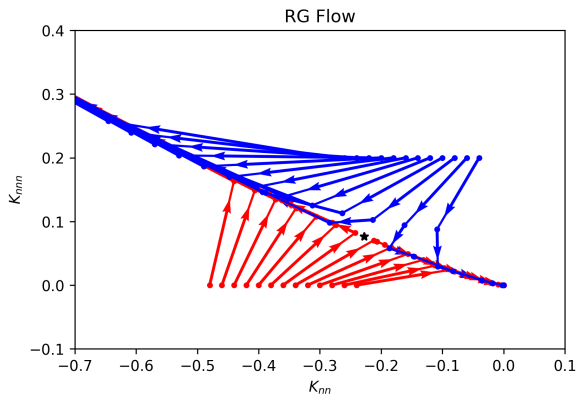


FIG. 18: The RG flow of several points in the region $K_{nn} < 0$ and $K_{nnn} > 0$. The critical fixed point is marked as a black star.

In Figure 18, we analyze the criticality of the system in another region where $K_{nn} < 0$ and $K_{nnn} > 0$. The critical point for this case is found as: $K_{nn} = -0.227$ and $K_{nnn} = 0.076$. On the right hand side, there is the high temperature fixed point at the origin that we have mentioned before. On the left hand side, the points are flowing to the low temperature fixed point that indicates the antiferromagnetic order in the lattice due to the strong next nearest neighbor interaction and negative nearest neighbor interaction constant ($K_{nn} \ll 0$ and $K_{nnn} \gg 0$). For the critical fixed points, the correlation length becomes $\xi = \infty$ and for the other fixed points which are defined as trivial we have that $\xi = 0$ [6].

4. Conclusion

Overall, in this work we have utilized two different approaches for solving the Ising model. The first method, Mean Field approximation, has allowed us to work with 2 different lattice types (wavy and antiferromagnetic) separately. Constructing four possible states for a two-spin system, we have found the phase transition lines and critical surfaces of both lattice types. For this case, we also analyzed the critical behaviour of the lattice as the two heat baths had different temperatures. Since there are two processes involved in the spin-flip dynamics, one where only a single spin flips and the other where the spins are exchanged in between the two particles, the second dynamic is defined to be governed by the second heat bath for our model. These two separate dynamics allowed us to analyze more complicated behavior of the system in terms of its criticality as each is governed by different heat baths. In both lattice types, the addition of two different temperatures, has moved the phase transition lines directly.

The second method we used was the renormalization group (RG) transformations. For this case, the scaling model is constructed such that it is valid for both the antiferromagnetic and wavy lattices. Therefore, the result that shows the criticality of the system could span and read both phases of the lattice as each phase has its own trivial fixed point. The map of the consecutive coupling constants, K_{nn} and K_{nnn} , after each iteration is plotted. The behaviour of these maps have shown to be directed to the ferromagnetic and antiferromagnetic fixed points, while deflecting from the critical fixed points. The critical exponent " ν ", which is a universal value determining the criticality of many models in statistical physics, is found to be nearly 0.74 in this work.

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5. Appendix

```

1 # -*- coding: utf-8 -*-
2 """
3 Created on Tue Nov 21 12:57:31 2023
4
5 @author: Hp
6 """
7 import numpy as np
8 import matplotlib.pyplot as plt
9 import scipy as sp
10 from scipy.integrate import odeint
11 from scipy.integrate import solve_ivp
12 from mpl_toolkits import mplot3d
13 from matplotlib import cm
14 from matplotlib.ticker import LinearLocator
15 from mpl_toolkits.mplot3d import Axes3D
16
17 def calculate_mag(a,b,K,G, T_ratio):
18
19     def w12(x,y):
20         return(a*(1 - np.tanh(K*(3*x + 1) + 4*G*y)))
21
22     def w21(x,y):
23         return(a*(1 + np.tanh(K*(3*x + 1) + 4*G*y)))
24
25     def w13(x,y):
26         return(a*(1 - np.tanh(K*(3*y + 1) + 4*G*x)))
27
28     def w31(x,y):
29         return(a*(1 + np.tanh(K*(3*y + 1) + 4*G*x)))
30
31     def w24(x,y):
32         return(a*(1 - np.tanh(K*(3*y - 1) + 4*G*x)))
33
34     def w42(x,y):
35         return(a*(1 + np.tanh(K*(3*y - 1) + 4*G*x)))
36
37     def w34(x,y):
38         return(a*(1 - np.tanh(K*(3*x - 1) + 4*G*y)))
39
40     def w43(x,y):
41         return(a*(1 + np.tanh(K*(3*x - 1) + 4*G*y)))
42
43     def w23(x,y):

```

```

44     return(b*(1 - np.tanh(T_ratio*(3*K-4*G)*(y-x))))
45
46 def w32(x,y):
47     return(b*(1 + np.tanh(T_ratio*(3*K-4*G)*(y-x))))
48
49
50 def dPdt(t,P):
51     P1, P2, P3, P4 = P
52     m1 = P1 + P2 - P3 - P4
53     m2 = P1 - P2 + P3 - P4
54
55     w12m = w12(m1,m2)
56     w21m = w21(m1,m2)
57     w13m = w13(m1,m2)
58     w31m = w31(m1,m2)
59     w24m = w24(m1,m2)
60     w42m = w42(m1,m2)
61     w34m = w34(m1,m2)
62     w43m = w43(m1,m2)
63     w23m = w23(m1,m2)
64     w32m = w32(m1,m2)
65
66     term1 = -1 * (w12m+w13m) * P1 + w21m * P2 + w31m * P3
67     term2 = w12m * P1 - (w21m+w23m+w24m) * P2 + w32m * P3 + w42m * P4
68     term3 = w13m * P1 + w23m * P2 - (w31m+w32m+w34m) * P3 + w43m * P4
69     term4 = w24m*P2 + w34m * P3 - 1 * (w42m+w43m) * P4
70
71     return [term1, term2, term3, term4]
72
73 P1_0= 0.3
74 P2_0= 0.3
75 P3_0= 0.2
76 P4_0= 0.2
77
78 P_0= (P1_0, P2_0, P3_0, P4_0)
79
80 t = np.linspace(0, 8, 100)
81 soln = odeint(dPdt, y0=P_0, t=t, tfirst=True)
82
83 P1_soln = soln.T[0]
84 P2_soln = soln.T[1]
85 P3_soln = soln.T[2]
86 P4_soln = soln.T[3]
87
88 mag1 = P1_soln+P2_soln-P3_soln-P4_soln
89 mag2 = P1_soln-P2_soln+P3_soln-P4_soln
90 mag1_eq = mag1[-1]
91 mag2_eq = mag2[-1]
92 return(mag1_eq,mag2_eq)
93
94
95 K_val = np.linspace(-0.9,0.9,50)
96 G_val = np.linspace(-0.9,0.9,50)
97
98 K_mesh, G_mesh = np.meshgrid(K_val, G_val)
99 mag1 = np.zeros_like(K_mesh)

```

```

100 mag2 = np.zeros_like(K_mesh)
101 product = np.zeros_like(K_mesh)
102
103
104 for i in range(len(K_val)):
105     for j in range(len(G_val)):
106         K = K_val[i]
107         G = G_val[j]
108         mags = calculate_mag(2, 2, K, G, 0.2)
109         #print(mags[1])
110         mag1[j][i] = mags[0]
111         mag2[j][i] = mags[1]
112         product[j][i]=mags[0]*mags[1]
113
114
115 fig = plt.figure(figsize=(9, 6), dpi=300)
116 # Subplot 1
117 ax1 = fig.add_subplot(1, 2, 1, projection='3d')
118 surf1 = ax1.plot_surface(K_mesh, G_mesh, mag1, cmap='viridis')
119 ax1.set_xlabel('K values')
120 ax1.set_ylabel('G values')
121 ax1.set_title('$m_1$')
122 ax1.view_init(elev=50, azim=170)
123 cbar1 = fig.colorbar(surf1, ax=ax1, shrink=0.5, aspect=10)
124 ax1.w_zaxis.set_major_formatter(plt.NullFormatter())
125
126 # Subplot 2
127 ax2 = fig.add_subplot(1, 2, 2, projection='3d')
128 surf2 = ax2.plot_surface(K_mesh, G_mesh, mag2, cmap='viridis')
129 ax2.set_xlabel('K values')
130 ax2.set_ylabel('G values')
131 ax2.set_title('$m_2$')
132 ax2.view_init(elev=50, azim=170)
133 cbar2 = fig.colorbar(surf2, ax=ax2, shrink=0.5, aspect=10)
134 ax2.w_zaxis.set_major_formatter(plt.NullFormatter())
135
136 plt.subplots_adjust(right=1, wspace=0.2)
137 plt.show()
138
139
140 fig1 = plt.figure(figsize=(8, 6), dpi=300)
141 ax = fig1.add_subplot(1, 1, 1, projection='3d')
142 surff = ax.plot_surface(K_mesh, G_mesh, product, cmap='viridis',vmin=-1, vmax=1)
143 ax.set_xlabel('K values',fontsize=16, labelpad=15)
144 ax.set_ylabel('G values',fontsize=16, labelpad=15)
145 ax.set_title('$m_1 \cdot m_2$',fontsize=20)
146 ax.view_init(elev=90, azim=0)
147 cbarr = fig1.colorbar(surff, ax=ax, shrink=0.8, aspect=10, ticks=[-1, 0, 1])
148 cbarr.ax.tick_params(labelsize=14)
149 ax.set_xticks([-0.5, 0, 0.5])
150 ax.set_yticks([-0.5, 0, 0.5])
151 ax.w_zaxis.set_major_formatter(plt.NullFormatter())
152
153
154 plt.show()

```

Listing 1: Python code for the antiferromagnetic lattice using the Mean Field approximation (see in Figure 12).

```

1 import numpy as np
2 import matplotlib.pyplot as plt
3 from statistics import mean
4
5
6 def plot_RG_flow(Knn_in,Knnn_in,iteration_number):
7
8     def RG(Knn,Knnn):
9         #summation over S1, S2, S3, S4 = +- 1
10        vals = [-1 ,1]
11        KnnP = 0
12        KnnnP = 0
13
14        for G1 in vals:
15            for G2 in vals:
16                for G3 in vals:
17                    for G4 in vals:
18                        f = 0
19                        for S1 in vals:
20                            for S2 in vals:
21                                for S3 in vals:
22                                    for S4 in vals:
23                                        f += np.exp((Knn*1.5)*(G1*S1 + S1*S2 + S4*S3 + S3*S3 + S1*
S4 + S2*S3 + S2*S2 + S4*S4) + (Knnn*7/3)*(S1*S2 + S2*S4 + S3*S4 + G1*S4 + S1*S3 + S2*S3))
24
25
26                            KnnP += G1*S2* np.log(f)
27                            KnnnP += G1*S3*np.log(f)
28
29        KnnP = KnnP/8
30        KnnnP = KnnnP/16
31        #print("for the initial values Knn = ",Knn,"and Knnn = ",Knnn,"the scaled constants are Knn
' = ",KnnP,"and KnnnP' = ",KnnnP)
32        return[KnnP, KnnnP]
33
34
35 Knn_points = [Knn_in]
36 Knnn_points = [Knnn_in]
37
38 for i in range(0,iteration_number):
39     [Knn_p, Knnn_p] = RG(Knn_in, Knnn_in)
40
41     Knn_in = Knn_p
42     Knnn_in = Knnn_p
43
44     Knn_points.append(Knn_p)
45     Knnn_points.append(Knnn_p)
46
47     return(Knn_points, Knnn_points)
48
49
50
51 func = plot_RG_flow(-0.342, 0, 5)
52 Knn = func[0]
53 Knnn = func[1]
54

```

```

55 K_A = np.array([Knn[1],Knnn[1]])
56 K_B = np.array([Knn[2],Knnn[2]])
57 K_C = np.array([Knn[3],Knnn[3]])
58 lamd = (K_C-K_B)/(K_B-K_A) + 1
59
60
61 b=np.sqrt(5)
62 exponent = np.emath.logn(b, lamd)
63 Nu = 1/exponent
64 print(Nu)
65
66
67 plt.figure(dpi=300)
68 plt.xlabel('$K_{nn}$')
69 plt.ylabel('$K_{nnn}$')
70 plt.xlim(-0.7,0.1)
71 plt.ylim(-0.1,0.4)
72 plt.title('RG Flow')
73
74 vals = [0,1,2]
75 for h in [-0.08, -0.06, -0.04, -0.02, 0, 0.02, 0.04, 0.06, 0.08, 0.1, 0.12, 0.14, 0.16]:
76     func = plot_RG_flow(-0.4 +h, 0., 5)
77
78     if h == 0.04:
79         Knn10 = func[0][1]
80         Knnn10 = func[1][1]
81     if h == 0.06:
82         Knn11 = func[0][1]
83         Knnn11 = func[1][1]
84
85
86     Knn = func[0]
87     Knnn = func[1]
88
89
90     K_A = np.array([Knn[1], Knnn[1]])
91     K_B = np.array([Knn[2], Knnn[2]])
92     K_C = np.array([Knn[3], Knnn[3]])
93
94     plt.plot(Knn, Knnn, c='r', marker='.', label='Points')
95
96
97     for i in vals:
98         plt.quiver(Knn[i], Knnn[i], Knn[i + 1] - Knn[i], Knnn[i + 1] - Knnn[i],
99                 angles='xy', scale_units='xy', scale=1.2, color='r', width=0.006)
100
101 for h in [-0.08, -0.06, -0.04, -0.02, 0, 0.02, 0.04, 0.06, 0.08, 0.1, 0.12, 0.14, 0.16]:
102     func = plot_RG_flow(-0.2+h, 0.2, 5)
103
104     if h == -0.02:
105         Knn20 = func[0][1]
106         Knnn20 = func[1][1]
107     if h == 0:
108         Knn21 = func[0][1]
109         Knnn21 = func[1][1]
110

```



```

111 Knn = func[0]
112 Knnn = func[1]
113 #print(Knn,Knnn)
114
115 K_A = np.array([Knn[1], Knnn[1]])
116 K_B = np.array([Knn[2], Knnn[2]])
117 K_C = np.array([Knn[3], Knnn[3]])
118
119 plt.plot(Knn, Knnn, c='b', marker='.', label='Points')
120
121 for i in vals:
122     plt.quiver(Knn[i], Knnn[i], Knn[i + 1] - Knn[i], Knnn[i + 1] - Knnn[i],
123              angles='xy', scale_units='xy', scale=1.2, color='b', width=0.006)
124
125 fp_Knn = mean([Knn10,Knn11])
126 fp_Knnn = mean([Knnn10,Knnn11])
127 plt.plot(fp_Knn, fp_Knnn, c='k', marker='*', label = 'The fixed point')
128 print('The critical fixed point is at ', fp_Knn, fp_Knnn)
129 plt.show()

```

Listing 2: Python code for the solution by the RG transformations which can be seen in Figure 18.