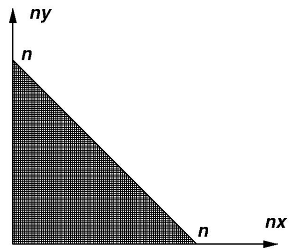
1.



(a) $E = \hbar\omega_c(n_x + n_y + 1)$ with $n_x, n_y = 0, 1, 2, \cdots$.

(b) All particles with same $n = n_x + n_y$ have the same energy. That equation corresponds to the diagonal line in the figure. There are $2 \times \text{Area}/(1 \times 1) = n^2$ states inside the triangle. So, when $E_F = \hbar\omega_c(n+1)$, there are $N = n^2$ occupied states. So, $E_F = \hbar\omega_c(\sqrt{N} + 1)$.

(c) There are $2ndn$ states between n and $n + dn$. So, the total energy becomes

$$U = \int_0^{\sqrt{N}} \hbar\omega_c(n+1)2ndn \approx \frac{2}{3}\hbar\omega_c N^{3/2}.$$

2.(a) Ordering the total angular momentum states in the sequence $|00\rangle$ $|11\rangle$ $|10\rangle$ $|1-1\rangle$,

state	probability	expansion	vector
$\|\uparrow\uparrow\rangle$	1/4	$\|1\ 1\rangle$	$(0\ 1\ 0\ 0)^T$
$(\ \|\uparrow\uparrow\rangle + i\,\|\downarrow\downarrow\rangle\ )/\sqrt{2}$	1/4	$(\ \|1\ 1\rangle + i\,\|1\ -1\rangle\ )/\sqrt{2}$	$\left(0\ \frac{1}{\sqrt{2}}\ 0\ \frac{i}{\sqrt{2}}\right)^T$
$(\ \|\uparrow\downarrow\rangle - \|\downarrow\uparrow\rangle\ )/\sqrt{2}$	1/2	$\|0\ 0\rangle$	$(1\ 0\ 0\ 0)^T$

(b)

$$\rho = \frac{1}{4}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}(0\ 1\ 0\ 0) + \frac{1}{4}\begin{pmatrix}0\\ \frac{1}{\sqrt{2}}\\0\\ \frac{i}{\sqrt{2}}\end{pmatrix}\left(0\ \frac{1}{\sqrt{2}}\ 0\ \frac{-i}{\sqrt{2}}\right) + \frac{1}{2}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}(1\ 0\ 0\ 0) = \begin{pmatrix}\frac{1}{2} & 0 & 0 & 0\\ 0 & \frac{3}{8} & 0 & \frac{-i}{8}\\ 0 & 0 & 0 & 0\\ 0 & \frac{i}{8} & 0 & \frac{1}{8}\end{pmatrix}$$

Easy to check that $\text{Tr}(\rho) = 1$.

(c)

$$[L^2] = \text{Tr}(\rho L^2) = \text{Tr}\left[\begin{pmatrix}\frac{1}{2} & 0 & 0 & 0\\ 0 & \frac{3}{8} & 0 & \frac{-i}{8}\\ 0 & 0 & 0 & 0\\ 0 & \frac{i}{8} & 0 & \frac{1}{8}\end{pmatrix}\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 2\end{pmatrix}\hbar^2\right] = \text{Tr}\begin{pmatrix}0 & \cdot & \cdot & \cdot\\ \cdot & \frac{3}{4} & \cdot & \cdot\\ \cdot & \cdot & 0 & \cdot\\ \cdot & \cdot & \cdot & \frac{1}{4}\end{pmatrix}\hbar^2 = \hbar^2$$

(d)

$$[L_z] = \text{Tr}(\rho L_z) = \text{Tr}\left[\begin{pmatrix}\frac{1}{2} & 0 & 0 & 0\\ 0 & \frac{3}{8} & 0 & \frac{-i}{8}\\ 0 & 0 & 0 & 0\\ 0 & \frac{i}{8} & 0 & \frac{1}{8}\end{pmatrix}\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\hbar\right] = \text{Tr}\begin{pmatrix}0 & \cdot & \cdot & \cdot\\ \cdot & \frac{3}{8} & \cdot & \cdot\\ \cdot & \cdot & 0 & \cdot\\ \cdot & \cdot & \cdot & \frac{-1}{8}\end{pmatrix}\hbar = \frac{\hbar}{4}$$

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3. (a) $\iiint |R_{10}(r)Y_1^0(\theta,\phi)|^2 \sin\theta\, d\theta d\phi r^2 dr = A^2\int_0^\infty \exp(-2r/a)r^2dr = A^2\ 2!/(2/a)^3 \Longrightarrow A = 2a^{-3/2}$

(b) $E_{10}^{(1)} = A^2\int_0^\infty V_o \exp(-2r/a - r/D)r^2dr = V_oA^2\ 2!/[(2/a) + 1/D]^3 = V_o/(1 + a/2D)^3$

(c) When $D \to \infty$, one is left with a constant potential V_o, which trivially shifts all energies by that amount.