Homework 6[1]

A total charge Q is uniformly distributed over the surface of a sphere of radius R. The sphere rotates about the z-axis with angular velocity ω. The magnetic field of this construction has been calculated, and is known to be

$$\vec{B} = \begin{cases} \vec{B}_{in} = b_o \omega \hat{z} & \text{if } r < R, \\ \vec{B}_{out} = (b_o \omega R^3/2r^3)\,(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}) & \text{if } r > R \end{cases}$$

where

$$b_o = \frac{\mu_o}{4\pi}\,\frac{2}{3}\,\frac{Q}{R}\,.$$

(a) Calculate the electric field $\vec{E}$ for this charge density.

(b) Find the Poynting vector $\vec{S} = \vec{E} \times \vec{B}/\mu_o$ at all points in space.

(c) Consider the case where $d\omega/dt = \alpha \neq 0$. Calculate the Faraday induced electric field (magnitude and direction) at the surface of the sphere as a function of θ.

(d) Calculate the torque this induced electric field produces on the sphere.

(e) Remember that the quantity $\vec{\wp} = \mu_o \epsilon_o \vec{S}$ is the momentum density in space. From the symmetry of this quantity, can you guess what the total momentum imparted to the accelerating sphere will be? Why?

(f) The quantity $\vec{\ell} = \vec{r} \times \vec{\wp}$ is the angular momentum density. From the symmetry of $\vec{\ell}$ can you guess what the direction of the total angular momentum imparted to the accelerating sphere will be? Why? (Result of part (d) should be a hint.)

(g) Integrate this component of $\vec{\ell}$ to find the total angular momentum being imparted to the sphere and show that it is consistent with the result of part (d).

[1] This problem was inspired by an exam question in the MIT Open Courseware: https://ocw.mit.edu/courses/8-07-electromagnetism-ii-fall-2012/resources/mit8_07f12_finalexam/