Final Exam Solutions — Bilkent University

Fall 2023 — Phys 315

1.(a) The image will be at $z=-a$, and will have the same magnitude, and will also point in the $+z$ direction.
(b) Since $\vec{E}=(3(\vec{p}\cdot\hat{r})\hat{r}-\vec{p})/4\pi\epsilon_o r^3$, we will have $\vec{E}(z)=2p_o\hat{z}/[4\pi\epsilon_o(z+a)^3]$.
(c) Since $\vec{E}$ and $\vec{p}$ are in the same direction, torque on the dipole will be zero.
(d) Since $\vec{F}=(\vec{p}\cdot\vec{\nabla})\vec{E}$, we have $\vec{F}=p_o\left(\partial/\partial z\ \vec{E}\right)_{(0,0,a)}=-\hat{z}3p_o^2/(32\pi\epsilon_o a^4)$.
The x and y components of the field are proportional to x and y respectively and do not contribute to the force at $(0,0,a)$.

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2. (a) $\rho_{\text{bound}}=-\nabla\cdot\vec{\mathrm{P}}=-\partial/\partial z\ \alpha z=-\alpha$.
(b) $\sigma_{\text{bound}}=\vec{\mathrm{P}}\cdot\hat{r}|_S=\mathrm{P}(R,\theta)\ \hat{z}\cdot\hat{r}=\alpha\underbrace{R\cos(\theta)}_{z}\cdot\underbrace{\cos(\theta)}_{\hat{z}\cdot\hat{r}}$
(c) Since $P_2(x)=(3x^2-1)/2$, we have $x^2=(2P_2(x)+1)/3$
so that $\sigma_{\text{bound}}=\alpha R\cos^2(\theta)=\alpha R[2P_2(\cos\theta)+P_0(\cos\theta)]/3$.
(d) We will have only $l=0$ and $l=2$ contributions:

$$V_{\text{in}}(r,\theta)=A_0+A_2r^2P_2(\cos\theta)\qquad V_{\text{out}}(r,\theta)=B_0/r+B_2P_2(\cos\theta)/r^3$$

with the boundary conditions

$$V_{\text{in}}(r,\theta)|_{r=R}=V_{\text{out}}(r,\theta)|_{r=R}\qquad\text{and}\qquad\underbrace{\left(-\frac{\partial}{\partial r}V_{\text{out}}(r,\theta)\right)_{r=R}}_{\hat{r}\cdot E_{\text{out}}}-\underbrace{\left(-\frac{\partial}{\partial r}V_{\text{in}}(r,\theta)\right)_{r=R}}_{\hat{r}\cdot E_{\text{in}}}=\sigma/\epsilon_o$$

which results in

$$\begin{array}{ll} A_0=B_0/R & A_2R^2=B_2/R^3 \\ B_0/R^2=\alpha R/3\epsilon_o & 3B_2/R^4+2A_2R=2\alpha R/3\epsilon_o \end{array}$$

so that $B_0=\alpha R^3/3\epsilon_o$, $A_0=\alpha R^2/3\epsilon_o$, $B_2=2\alpha R^5/15\epsilon_o$ and $A_2=2\alpha/15\epsilon_o$

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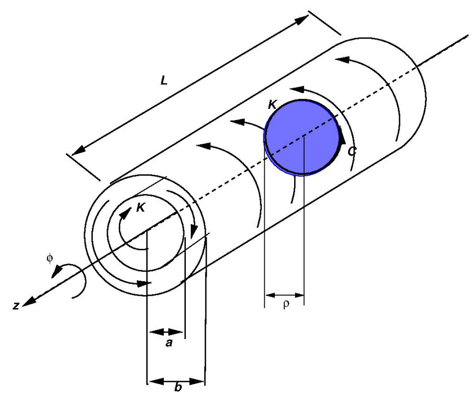
3. (a) Using superposition, and noting that the selenoidal current produces a magnetic field only inside the cylinder, ve obtain the field structure $\vec{B}=\hat{z}\mu_o K$ for $a<\rho<b$ and zero elsewhere.
(b) Applying $\oint_c\vec{A}\cdot d\vec{l}=\int_S\vec{B}\cdot d\vec{S}$ to the blue surface S (bounded by the co-axial circle C with radius ρ) in the figure,



$$\oint_c\vec{A}\cdot d\vec{l}=A_\phi 2\pi\rho=\begin{cases}0 & \text{for } \rho<a\\ B\pi(\rho^2-a^2) & \text{for } a<\rho<b\\ B\pi(b^2-a^2) & \text{for } \rho>b\end{cases}$$

(c) Noticing that $\vec{A}$ is in the ϕ direction, and a function of ρ only, the only term that will give a finite contribution is $\vec{B}=\hat{z}\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\phi)$. From the form of ρA_ϕ it is quite apparent that one will get zero unless $a<\rho<b$, and the differentiation will then give $\vec{B}$ found in part (a).