1.Potential at large ρ due to the electric field will be -$E_o x = -E_o \rho \cos\phi$. We have solutions of type

$$V(\rho,\phi) = -E_o \rho \cos\phi + \sum_m A_m \rho^{-m} \cos(m\phi)$$

outside of the cylinder. Other ρ^{+m} and $\ln\rho$ solutions are not appropriate because they would not satisfy the BC at large ρ, and $\sin(m\phi)$ solutions are not present because they are odd in ϕ and the source (the external field) produces only an even term in ϕ.
The BC at $\rho = R$ results in $E_o R \cos\phi + \sum_m A_m R^{-m} \cos(m\phi) = 0$ so that we have

$$\begin{array}{lll} -E_o + A_1/R = 0 & \text{for } m = 1 & \\ A_m/R^m = 0 & \text{for } m > 1 & \Longrightarrow \quad \text{all } A_m = 0 \text{ for } m > 1. \end{array}$$

So, the solution for the potential is $V(\rho,\phi) = E_o(R^2/\rho - \rho)\cos\phi$.

2.See first exam solutions.

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3. Remember that $1/(1+\epsilon) = 1 - \epsilon + \epsilon^2 - \epsilon^3 + \epsilon^4 \cdots$.

$$\begin{array}{ll}
\text{(a) } V(z) & = \frac{q}{4\pi\epsilon_o}\left(-\frac{2}{z} + \frac{1}{z+a} + \frac{1}{z-a}\right) \\
\text{(b)} & = \frac{q}{4\pi\epsilon_o z}\left(-2 + \frac{1}{1+a/z} + \frac{1}{1-a/z}\right) \\
 & = \frac{q}{4\pi\epsilon_o z}\left(-2 + [1 - \frac{a}{z} + (\frac{a}{z})^2 - (\frac{a}{z})^3 + (\frac{a}{z})^4 + \cdots] + [1 + \frac{a}{z} + (\frac{a}{z})^2 + (\frac{a}{z})^3 + (\frac{a}{z})^4 + \cdots]\right) \\
 & = \frac{q}{2\pi\epsilon_o z}\left((\frac{a}{z})^2 + (\frac{a}{z})^4 + \cdots\right) \\
\text{(c) } V(r,\theta) & = \frac{q}{2\pi\epsilon_o r}\left((\frac{a}{r})^2 P_2(\cos\theta) + (\frac{a}{r})^4 P_4(\cos\theta) + \cdots\right)
\end{array}$$

Note that the first term in the expansion corresponds to the quadruple term.