DIMENSIONAL ANALYSIS

When working through a complicated derivation, it is important to be sure that the units on one side of the resulting equation are the same as those on the other side. For example, in a calculation of the distance travelled by an object, one could be certain that a mistake had been made if the result came out in units of mass. An analysis of this sort is usually called *dimensional analysis* - a technique used in the physical sciences and engineering to reduce physical properties such as acceleration, viscosity, energy, and others to their fundamental dimensions of length, mass, time and charge. Whether the actual units of the fundamental dimensions are in the cgs or mks system is irrelevant. This technique facilitates the study of interrelationships of systems and their properties, and avoids the nuisance of incompatible units.

As an example of the use of dimensional analysis, suppose one reaches an equation of force

$$f = \frac{3}{5}\rho v^2$$

where ρ is the mass per unit volume, or density, and v is the speed. Dimensional analysis will never tell whether the factor 3/5 is correct since it is dimensionless, that is, a pure number. However, let us see whether ρv^2 has indeed dimensions of force. Using M, L, and T to denote mass, length, and time, we obtain

$$\rho = [M][L]^{-3}\ ,\ \ v^2 = [L]^2[T]^{-2}$$

and hence

$$\rho v^2 = [M][L]^{-3}[L]^2[T]^{-2} = [M][L]^{-1}[T]^{-2}.$$

On the other hand, force is mass times acceleration, acceleration is velocity divided by time, and velocity is length per time; so that

$$F = [M][L][T]^{-2}.$$

We thus reach the conclusion that there should be some mistake in the derivation since the dimensions on one side of the force equation are not consistent with the dimensions on the other side.

As a second illustration of the use of dimensional analysis, consider the case of a spherical body moving slowly through a viscous medium such as oil. In such a case, the damping force opposing the motion is governed by Stoke's law

$$f_{damp} = -6\pi\eta r v\ ,$$

where η is the coefficient of viscosity of the medium, r is the radius of the sphere, and v is the velocity. Using dimensional analysis, the dimensions of the coefficient of viscosity can readily be achieved through the relation $\eta \sim f/(rv)$, i.e.,

$$\eta = \frac{[f]}{[L][v]} = \frac{[M][L][T]^{-2}}{[L][L][T]^{-1}} = [M][L]^{-1}[T]^{-1}.$$

MEASUREMENT

Measurement is very important in the physical world. During observations of natural phenomena, we make measurements of various kinds. Measurement is an operation which assigns a value to a physical quantity in a selected unit system. In nature there are seven basic units. Unit of length (meter), of mass (kilogram), of time (second), and of the electronic charge (coulomb) are four of these basic units. All other units are combinations of these basic units. The full list of basic units and their relations with the derived units is tabulated on the cover of this manual.

Measurements can be classified in two groups: In *direct measurements*, the quantity measured is compared with a known or standard quantity by the use of a measuring instrument. In *indirect measurements*, the measured quantity is calculated by using results of some direct measurements. For example, speed of a particle is calculated from the measured distance taken on a specified time interval.

ERRORS

Result of a measurement depends on measuring instruments and the observer. It is not possible to perform measurements exactly. Therefore, it is necessary to include an error term in measurements. Error is not a mistake in the measurement, but is the uncertainty in the measurement. It indicates how close the measured quantity is to its exact value. Closely related with the notion of error are two concepts called *precision* and *accuracy*.

Precision of a measurement shows the reproducibility of the measurement, expressing the deviation from the average of many measurements using the same procedure. It is the degree of consistency and agreement among independent measurements of the same quantity, and also is the reliability or reproducibility of the result.

Accuracy is the closeness of agreement between a measured value and a true or accepted value. Accuracy of a measuring tool is related with how well the measuring tool is calibrated. For example, one can measure length with one millimeter accuracy by using a ruler, however, it is possible to perform measurements within 0.05 millimeter accuracy by using a micrometer.

The statement of uncertainty associated with a measurement should include factors that affect both the precision and accuracy of the measurement.

Types of Errors

Errors can be classified into two groups according to their nature as *systematic* and *random* errors.

Systematic Errors:

Systematic errors are those which arise from the measurement system and they are same for different measurements under same conditions. Main source of these type of errors is the calibration of measuring instruments. For example, if the zero setting of a voltmeter is shifted, all voltage readings are also shifted in a uni-directional manner, either all positive or all negative.

Systematic errors must be eliminated, or at least reduced to a negligible magnitude.

Random Errors:

Remaining type of errors are random errors, which are due to random fluctuations in the experimental situation. They are bidirectional, meaning that they can be either positive or negative. For example, to measure the length of a line segment AB, it is necessary to align the zero line of the ruler to coincide exactly with point A, but one can never be sure about the perfect alignment. Moreover, interpolation of the position of the end point B is also uncertain due to estimation of position and the parallax. Parallax is an optical error that arises when a meter is read from one side rather than straight on. Random errors can be treated by statistical methods. If one makes the measurement many times and then calculates the average of the independent measurements, random errors will be reduced, since the positive and negative contributions will cancel each other.

Estimation of Errors

Since the result of a measurement cannot be exact, it is important to estimate the error in recording a measurement. Error estimates help one to know how close the measurements are to the exact value.

The upper band of error can be equal to the least count or to the smallest readable scale division of the instrument. For example, in a measurement with a ruler in which the smallest division is one millimetre, maximum possible error is one millimeter.

One can make a better estimation using this ruler, such as ± 0.5 mm, but ± 0.2 mm is an overestimation since the naked eye can not resolve such a small quantity.

Recording Measurements

Recording a quantity in a measurement consists of the measured value, the uncertainty in it, and, obviously, the unit. Generally, the last digit in the measured value is the estimated one. Then the error term follows. If the error is bidirectional, $\pm$ sign is used. Otherwise, only the proper sign is included. Finally, the unit of the measurement is given ($x \pm \Delta x$ *unit*). Consider the length of a line segment measured by a ruler with one millimeter smallest division. We write $L = 10.65 \pm 0.05$ cm, where 10.6 is measured exactly and the last digit (5) is estimated. Then follows the error and the unit. The meaning of this result is that the value of L is somewhere between 10.6 cm and 10.7 cm.

Significant Figures

Only those figures or digits of a numerical quantity which are the result of an actual measurement, or calculation from an actual measurement are said to be significant. That is, digits including the last one, which is the estimated one, are the significant ones. One should avoid using extra figures.

One must be careful about the figure 0 (zero), for it may or may not be significant. If it serves only to place the decimal point, i.e., 0.0017m, or in converting units, i.e., 1700μm, it is not significant. If it is the result of an actual measurement, i.e., 1.70 mm, it is significant. Whether the zero is significant or not can be clarified by using scientific notation. In scientific notation, quantities are recorded by using powers of 10 and a prefactor having one nonzero digit in front of the decimal point. All digits placed in front of the power of ten should be significant figures. For example, if a measurement of a length $110cm$ is expressed as $1.10 \times 10^2 cm$, this measurement has three significant figures, whereas if it is expressed as $1.100 \times 10^2 cm$, it has four significant figures.

Calculated quantities must also have the same number of significant figures as the operands. When adding or subtracting, it is useless to keep any more decimal places than are present in the number having the fewest decimal places (least significant figure). For example,

$$
\begin{array}{l}
31.3\overline{2} \\
0.0513\overline{8} \\
3.\overline{5} \\
+ \\
\hline
34.\overline{87}\,\overline{138}
\end{array}
$$

where a bar over a digit shows the estimated figures. Considering the doubtful figures, the result is given as $34.\bar{9}$. Or, one can directly write

$$
\begin{array}{r}
31.\bar{3} \\
0.\bar{1} \\
3.\bar{5} \\
+ \quad \\
\hline
34.\bar{9}
\end{array}
$$

When multiplying or dividing, the result should have the same number of significant figures as the term with the fewest significant figures. For example:

$$
\begin{array}{l}
1.25\bar{4} \\
2.\bar{4} \\
\times \\
\hline
\ \ \bar{5}\bar{0}\bar{1}\bar{6} \\
250\bar{8} \\
+ \\
\hline
3.\bar{0}\bar{0}\bar{9}\bar{6}
\end{array}
$$

So, the result is $3.\bar{0}$.

Expressing Error

The error in the measured quantity can be expressed in two ways:

i) Absolute Error: Estimated error can be given in an absolute scale. Then, it has the same unit as the measured quantity. So, the result of the measurement is expressed as

$$x \pm \Delta x \ \ \textit{unit}.$$

ii) Percentage Error: Another way is to use the ratio to give the relative error with respect to the measured quantity. Because of the ratio, percentage error is unitless.

$$x \ unit \pm \frac{\Delta x}{x}(100).$$

Operations with Measured Quantities

Addition and Subtraction: Assuming that we are going to add the quantities $x \pm \Delta x$ and $y \pm \Delta y$. So, we have to evaluate

$$R \pm \Delta R = (x \pm \Delta x) + (y \pm \Delta y).$$

Maximum possible error results if both of the added quantities contribute in the same direction, i.e.,

$$R + \Delta R = x + y + (\Delta x + \Delta y)$$

or

$$R - \Delta R = x + y - (\Delta x + \Delta y).$$

Similar arguments are valid for the subtraction. Therefore, in the result of addition or subtraction, absolute errors are added.

$$R \pm \Delta R = (x + y) \pm (\Delta x + \Delta y).$$

Multiplication and Division: Assume that *R* is the result of the multiplication of $x \pm \Delta x$ and $y \pm \Delta y$. Let us first examine multiplication.

$$R \pm \Delta R = (x \pm \Delta x)(y \pm \Delta y)$$

Considering the positive sign, we have

$$R + \Delta R = (x + \Delta x)(y + \Delta y) = xy\left(1 + \frac{\Delta x}{x} + \frac{\Delta y}{y} + \frac{\Delta x\, \Delta y}{xy}\right).$$

Neglecting the second order term $\frac{\Delta x\, \Delta y}{xy}$, which is much smaller than $\frac{\Delta x}{x}$ and $\frac{\Delta y}{y}$, we have

$$R + \Delta R = xy + xy\left(\frac{\Delta x}{x} + \frac{\Delta y}{y}\right),$$

meaning that the maximum percentage error is

$$\frac{\Delta R}{R} = \frac{\Delta x}{x} + \frac{\Delta y}{y}.$$

A similar argument gives

$$R - \Delta R = xy - xy\left(\frac{\Delta x}{x} + \frac{\Delta y}{y}\right)$$

Therefore, as a result of a multiplication, percentage errors are added and we have

$$R \pm \Delta R = xy \pm xy\left(\frac{\Delta x}{x} + \frac{\Delta y}{y}\right).$$

Writing the above expression in the form

$$R \pm \Delta R = xy \pm \left(\Delta x\, y + x\, \Delta y\right),$$

we see that if $R = x\,y$, then the absolute error in *R* is $\Delta R = \Delta x\, y + x\, \Delta y$, resembling the product rule where Δ behaves like the derivative.

Division is very similar to multiplication. Writing

$$R \pm \Delta R = \frac{x \pm \Delta x}{y \pm \Delta y} = \left(x \pm \Delta x\right)\left(y \pm \Delta y\right)^{-1},$$

and expanding

$$\left(y \pm \Delta y\right)^{-1} = \frac{1}{y}\left(1 \pm \frac{\Delta y}{y} + O(\Delta^2)\right)$$

and keeping only first order terms, we have

$$R \pm \Delta R = (x \pm \Delta x)\frac{1}{y}\left(1 \pm \frac{\Delta y}{y}\right) = \frac{x}{y} \pm \left(\frac{\Delta x}{y} + \frac{x\,\Delta y}{y^2}\right).$$

Writing the above result as

$$R \pm \Delta R = \frac{x}{y} \pm \frac{x}{y}\left(\frac{\Delta x}{x} + \frac{\Delta y}{y}\right),$$

we see that for multiplication and division, percentage error in the result is the sum of the percentage errors of the individual terms.

GRAPHS AND GRAPHICAL PROCEDURES

A common method of expressing a relation between two variable quantities is by a Cartesian graph (named after Rene Descartes). A Cartesian axis system consists of two mutually perpendicular lines usually called, respectively, the x and y axes. The coordinates of a point are obtained by projecting perpendicularly on these axes and assigning values by means of a scale on the axes. If f(x) is a function of *x,* then for a value of *x* there will be a y value, where *y = f(x),* and the function *f(x)* is graphed by marking the points with these coordinates, *x* and *y*. Such a graph permits the ready appreciation of certain characteristics of the function. Also a number of different functions can be readily compared by their graphs.

It is clear that nothing is essentially changed if the values that are marked on the axes are not proportional to the distances from the origin but more or less arbitrary scales are used. Points, the coordinates of which satisfy a given equation, can be plotted as before and a curve can be drawn from which corresponding values can be read. The form of the curve can be altered and in some cases simplified. A basic idea is to use such scales that the graphs of the equations under consideration become straight lines, which are easy to draw. For instance, the equation

$$af(x) + bg(y) + c = 0$$

that restricts a linear relationship in a function of x and a function of y in which a, b, c are constants, becomes a straight line in an XY plane, i.e.,

$$aX + bY + c = 0,$$

if the distances X and Y along the axes to the marks *x* and y are determined by the functions

$X = f(x)$ and $Y = g(y)$.

Well known examples based on this idea are the logarithmic and semi-logarithmic plots.
The former plots use the scales
$X = \log x$ and $Y = \log y$

and are convenient for plotting graphs of the relations of the form $y^m = ax^n$, because this may be written as

$$m \log y = n \log x + \log a$$

and the graph in the scales X and Y is a straight line.

The semi-logarithmic plots have scales such that one axis is linear and the other is logarithmic. They are useful in plotting the results of experiments in which one quantity is an exponential function of the other (m or n equal to 1).

A logarithmic scale can also be used to compare quantities of greatly varying size. An example of this is the line scale for the frequency of electromagnetic waves, in which the frequency of interest ranges from 1 Hertz to 10^{19} Hertz.

Notes on Graph Plotting:

- Give precise explanatory title to the graph.
- In plotting a graph, label the coordinates along each axis. Give quantity and units.
- Scale the axes so that the gathered data can be marked easily and the paper is used as much efficiently as possible. Do not use the data points as the scaling of the graph. They do not necessarily have to be on the scaled points all the time.
- Experimentally determined points can be located by using a dot. For each measured point the positive and negative maximum possible errors must be marked with error bars. In many experiments, the result of a set of data yields a straight-line graph when plotted. In practice, however, the measured points will be scattered because of the error they contain. Draw best and worst straight lines passing through the data points by observation.
- When more than one curve is drawn, it is desirable to distinguish between them by using different symbols, dotted or dashed lines.
- The process of matching an equation to a set of data points is called curve fitting. Curve fitting often requires the assumption of a certain type of equation, such as linear, power law, exponential etc. The principal questions that arise when fitting a curve are: (a) Should the curve pass through every point? or (b) Should it be drawn smoothly neat, but not necessarily through every point? Actually, very little is known of what occurs between points. When checking a law or other functional relation, there is usually a reason to suppose that a uniform curve (or straight line) will result. If one or even two points are quite far from the apparent curve, then one should check the experimental data to see if a mistake has been made. If none appears, the point may be, in general, disregarded.