

1. Consider the parametric curve

$$\mathcal{C}_q: \mathbf{r}(t) = (qt^3 + t + 4)\mathbf{i} + (qt^2 + t - 3)\mathbf{j} + (t^3 + t^2 + t)\mathbf{k}, \quad (-\infty < t < \infty),$$

where q is a constant.

a. Find all points where the curve $\mathcal{C}_5$ intersects the plane $x + 4y - 5z = 7$. [Note that in this part $q = 5$.]

$$\left. \begin{aligned} \vec{r}(t) &= (5t^3 + t + 4)\vec{i} + (5t^2 + t - 3)\vec{j} + (t^3 + t^2 + t)\vec{k} \\ x + 4y - 5z &= 7 \end{aligned} \right\}$$

$$\Rightarrow (5t^3 + t + 4) + 4 \cdot (5t^2 + t - 3) - 5 \cdot (t^3 + t^2 + t) = 7$$

$$\Rightarrow 15t^2 = 15 \Rightarrow t^2 = 1 \Rightarrow t = 1 \quad \text{or} \quad t = -1$$

$$\Downarrow \qquad \qquad \Downarrow$$

$$(x, y, z) = (10, 3, 3), \quad (-2, 1, -1)$$

b. Determine all values of the constant q for which the curve $\mathcal{C}_q$ lies in a plane, and for each of these values, find an equation for the corresponding plane.

$\mathcal{C}_q$ lies in the plane $ax + by + cz = d$ where a, b, c (not all 0) and d are constants

$$\Downarrow$$

$$a \cdot (qt^3 + t + 4) + b \cdot (qt^2 + t - 3) + c \cdot (t^3 + t^2 + t) = d \quad \text{for all } t$$

$$\Downarrow$$

$$(aq + c)t^3 + (bq + c)t^2 + (a + b + c)t + (4a - 3b - d) = 0 \quad \text{for all } t$$

$$\Downarrow$$

$$\left. \begin{aligned} \textcircled{1} \quad aq + c &= 0 \\ \textcircled{2} \quad bq + c &= 0 \\ \textcircled{3} \quad a + b + c &= 0 \\ \textcircled{4} \quad 4a - 3b - d &= 0 \end{aligned} \right\} \Rightarrow \left\{ \begin{aligned} \textcircled{1} - \textcircled{2} &\Rightarrow (a - b) \cdot q = 0 \Rightarrow (b = a \text{ or } q = 0) \text{ and} \\ \textcircled{1} + \textcircled{2} - 2 \times \textcircled{3} &\Rightarrow (a + b) \cdot (q - 2) = 0 \Rightarrow (b = -a \text{ or } q = 2) \end{aligned} \right\}$$

If $b = a$ and $b = -a$, then $a = b = 0 \xrightarrow{\textcircled{3}} c = 0$, contradiction.

If $q = 0$, then $b = -a \Rightarrow c = 0, d = 7a$.

The plane $x - y = 7$ contains $\mathcal{C}_0$.

If $q = 2$, then $b = a \Rightarrow c = -2a, d = a$.

The plane $x + y - 2z = 1$ contains $\mathcal{C}_2$.