2. Determine whether each of the following series converges or diverges.

a. $\displaystyle\sum_{n=3}^{\infty} \frac{(n-2)^n(n+1)^n}{n^{2n+1}}$

useful limits

$$c = \lim_{n\to\infty} \frac{\dfrac{(n-2)^n(n+1)^n}{n^{2n+1}}}{\dfrac{1}{n}} = \lim_{n\to\infty} \left(1-\frac{2}{n}\right)^n \cdot \left(1+\frac{1}{n}\right)^n = e^{-2} \cdot e = e^{-1}$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (the harmonic series) and

$c = e^{-1} > 0$, $\displaystyle\sum_{n=3}^{\infty} \frac{(n-2)^n(n+1)^n}{n^{2n+1}}$ diverges by Limit Comparison Test.

b. $\displaystyle\sum_{n=0}^{\infty} \frac{((2n)!)^2}{(3n)!\,n!}$

$$a_n = \frac{((2n)!)^2}{(3n)!\,n!}$$

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|\dfrac{((2\cdot(n+1))!)^2}{(3\cdot(n+1))!\,(n+1)!}\right|}{\left|\dfrac{((2n)!)^2}{(3n)!\,n!}\right|} = \lim_{n\to\infty} \left(\frac{(2n+2)!}{(2n)!}\right)^2 \cdot \frac{(3n)!}{(3n+3)!} \cdot \frac{n!}{(n+1)!}$$

$(2n+2)! = (2n+2)(2n+1)\cdot(2n)!$

$(3n+3)! = (3n+3)\cdot(3n+2)\cdot(3n+1)\cdot(3n)!$

$(n+1)! = (n+1)\cdot n!$

$$= \lim_{n\to\infty} \frac{((2n+2)(2n+1))^2}{(3n+3)(3n+2)(3n+1)(n+1)} = \frac{(2\cdot 2)^2}{3^3\cdot 1} = \frac{16}{27}$$

Since $L = \dfrac{16}{27} < 1$, $\displaystyle\sum_{n=0}^{\infty} \frac{((2n)!)^2}{(3n)!\,n!}$ converges by Ratio Test.