2. Suppose that the graph of a differentiable function $z = f(x, y)$ contains the parametric curves

$$C_1: \ \mathbf{r}_1 = (t+1)\,\mathbf{i} + (2t+1)\,\mathbf{j} + (2t^3 - 1)\,\mathbf{k}\,, \quad (-\infty < t < \infty)\,,$$

and

$$C_2: \ \mathbf{r}_2 = (2-t)\,\mathbf{i} + (3 - 4t + t^2)\,\mathbf{j} + (1 - t^2)\,\mathbf{k}\,, \quad (-\infty < t < \infty)\,.$$

a. Find an equation for the tangent plane to the graph of $z = f(x, y)$ at the point $(x, y, z) = (2, 3, 1)$.

$$\vec{v}_1 = \frac{d\vec{r}_1}{dt} = \vec{i} + 2\vec{j} + 6t^2\vec{k} \Rightarrow \vec{v}_1\Big|_{t=1} = \vec{i} + 2\vec{j} + 6\vec{k}$$

$$\vec{v}_2 = \frac{d\vec{r}_2}{dt} = -\vec{i} + (-4+2t)\vec{j} - 2t\vec{k} \Rightarrow \vec{v}_2\Big|_{t=0} = -\vec{i} - 4\vec{j}$$

$$\vec{n} = \vec{v}_1\Big|_{t=1} \times \vec{v}_2\Big|_{t=0} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 6 \\ -1 & -4 & 0 \end{vmatrix} = 24\vec{i} - 6\vec{j} - 2\vec{k}$$

An equation for the tangent plane is:

$$24 \cdot (x-2) + (-6) \cdot (y-3) + (-2) \cdot (z-1) = 0$$

OR

$$z = 12x - 3y - 14$$

b. Compute $\displaystyle \frac{d}{dt} f(6/t, 27/t^2)\Big|_{t=3}$.

$$\Rightarrow f_x(2,3) = 12\,, \quad f_y(2,3) = -3$$

$$\frac{d}{dt} f(6/t, 27/t^2) = f_x(6/t, 27/t^2) \cdot \frac{d}{dt}(6/t) + f_y(6/t, 27/t^2) \cdot \frac{d}{dt}(27/t^2)$$

$$= f_x(6/t, 27/t^2) \cdot (-6/t^2) + f_y(6/t, 27/t^2) \cdot (-54/t^3)$$

$$\Rightarrow \frac{d}{dt} f(6/t, 27/t^2)\Big|_{t=3} = f_x(2,3) \cdot \left(-\frac{2}{3}\right) + f_y(2,3) \cdot (-2) = 12 \cdot \left(-\frac{2}{3}\right) + (-3) \cdot (-2) = -2$$