4a. Find all x satisfying the equation $x - x^3 - x^4 + x^6 + x^7 - x^9 - x^{10} + \cdots \overset{\circledast}{=} -\frac{11}{111}$.

$$x-x^3-x^4+x^6+x^7-x^9-x^{10}+\cdots = (x-x^3)\cdot(1-x^3+x^6-x^9+\ldots) = \frac{x-x^3}{1+x^3} = \frac{x-x^2}{1-x+x^2}$$

if $|x|<1$, and the series diverges if $|x|\geq 1$.

If $|x|<1$, then:

$$\circledast \iff \frac{x-x^2}{1-x+x^2} = -\frac{11}{111} \iff 100x^2-100x-11=0 \iff x=\frac{11}{10} \text{ or } x=-\frac{1}{10}$$

Hence $x=-\frac{1}{10}$ is the only solution.

4b. Show that $\sum_{n=0}^{\infty} \frac{1}{n^2+4} < 1$.

$$\sum_{n=3}^{\infty}\frac{1}{n^2+4} < \int_2^{\infty}\frac{dx}{x^2+4} \underset{\boxed{\substack{x=2u\\dx=2du}}}{=} \frac{1}{2}\int_1^{\infty}\frac{du}{u^2+1} = \frac{1}{2}\lim_{c\to\infty}\int_1^c\frac{du}{u^2+1} = \frac{1}{2}\lim_{c\to\infty}\Big[\arctan u\Big]_1^c$$

$$= \frac{1}{2}\lim_{c\to\infty}(\arctan c - \arctan 1) = \frac{1}{2}\cdot\left(\frac{\pi}{2}-\frac{\pi}{4}\right) = \frac{\pi}{8}$$

$$\sum_{n=0}^{\infty}\frac{1}{n^2+4} = \frac{1}{4}+\frac{1}{5}+\frac{1}{8}+\sum_{n=3}^{\infty}\frac{1}{n^2+4} < \frac{1}{4}+\frac{1}{5}+\frac{1}{8}+\frac{\pi}{8} < \frac{1}{4}+\frac{1}{5}+\frac{1}{8}+\frac{2}{5} = \frac{39}{40} < 1$$