1. Let $f(x,y) = 2\cos(x+y) - 3 + e^{-(x^2+y^2)}$.

a. Evaluate the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{f(x,y)}$.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{f(x,y)} = \lim_{x\to 0} \frac{x\cdot 0}{f(x,0)} = \lim_{x\to 0} \frac{0}{2\cos x - 3 + e^{-x^2}} = \lim_{x\to 0} 0 = 0$$

along the x-axis

$$\lim_{(x,y)\to(0,0)} \frac{xy}{f(x,y)} = \lim_{x\to 0} \frac{x\cdot(-x)}{f(x,-x)} = \lim_{x\to 0} \frac{-x^2}{-1+e^{-2x^2}} \overset{\text{L'H}}{=} \lim_{x\to 0} \frac{-2x}{-4xe^{-2x^2}} = \frac{1}{2}$$

along the line $y=-x$

Since $0 \neq \frac{1}{2}$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{f(x,y)}$ does not exist by 2-Path Test.

b. $(0,0)$ is a critical point of f. [You do not need to verify this.] Determine whether f has a local maximum, a local minimum, a saddle point, or something else at $(0,0)$.

$$\begin{cases} \cos(x+y) \le 1 \text{ for all } (x,y) \\ -(x^2+y^2) \le 0 \Rightarrow e^{-(x^2+y^2)} \le 1 \text{ for all } (x,y) \end{cases}$$

$\Rightarrow f(x,y) = 2\cos(x+y) - 3 + e^{-(x^2+y^2)} \le 2 - 3 + 1 = 0 = f(0,0)$ for all (x,y)

$\Rightarrow f$ has its absolute maximum value at $(0,0)$

$\Rightarrow f$ has a local maximum value at $(0,0)$