Midterm Exam Question 5.

Consider a sequence $\{a_n\}_{n=1}^{\infty}$ satisfying the conditions

$$a_1 = A \qquad \text{and} \qquad a_{n+1} = \frac{3a_n^3}{a_n^2+4} \quad \text{for } n \geq 1$$

where A is a real number.

a. Suppose that the sequence $\{a_n\}$ converges, and let $L = \lim_{n\to\infty} a_n$. Show that, depending on A, there are at most three possible values for L.

b. Let A be your Bilkent student number. Determine whether the sequence $\{a_n\}$ converges or diverges.

Show all your work!
Explain your reasoning fully and in detail using correct mathematical notation and terminology, and in well-formed mathematical and English sentences!

(a) If $L = \lim_{n\to\infty} a_n$, then $a_{n+1} = \dfrac{3a_n^2}{a_n^2+4}$ for $n \geq 1$

$\Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \dfrac{3a_n^2}{a_n^2+4} \Rightarrow L = \dfrac{3L^3}{L^2+4}$

$\Rightarrow L = 0$ or $L^2+4 = 3L^2 \Rightarrow L=0$ or $L^2 = 2 \Rightarrow L = 0, \sqrt{2}$ or $-\sqrt{2}$.

(b) If $a_n \geq 2$ for some n, then $a_n^2 \geq 4 \Rightarrow 2a_n^2 \geq a_n^2 + 4$

$\Rightarrow 3a_n^2 \geq a_n^2 + 4 \Rightarrow \dfrac{3a_n^2}{a_n^2+4} \geq 1 \Rightarrow a_{n+1} = \dfrac{3a_n^3}{a_n^2+4} \geq a_n \geq 2$.

Since $a_1 = A \geq 2$; by induction, $a_n \geq 2$ for all $n \geq 1$.

Therefore, if $L = \lim_{n\to\infty} a_n$ exists, then $L \geq 2$.

But $L \leq \sqrt{2} < 2$ by part (a), a contradiction.

Hence the limit does not exist, and $\{a_n\}$ diverges.