*Do not forget to write your **full name** and your **Bilkent ID number**, and **sign** on the upper right corner of your paper.*

Midterm Exam Question 4.

Determine whether each of the following series converges or diverges:

a. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^6+2021}}$ **b.** $\displaystyle\sum_{n=0}^{\infty} \frac{((2n)!)^3}{(4n)!\,(n!)^2}$

Show all your work!
Explain your reasoning fully and in detail using correct mathematical notation and terminology, and in well-formed mathematical and English sentences!

(a)

$$c = \lim_{n\to\infty} \frac{\frac{n^2}{\sqrt{n^6+2021}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{2021}{n^6}}} = 1$$

Since $c = 1 > 0$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series), $\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^6+2021}}$ diverges by LCT.

(b) $a_n = \frac{((2n)!)^3}{(4n)!(n!)^2}$

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|\frac{((2n+2)!)^3}{(4n+4)!((n+1)!)^2}\right|}{\left|\frac{((2n)!)^3}{(4n)!(n!)^2}\right|} = \lim_{n\to\infty} \left(\frac{(2n+2)!}{(2n)!}\right)^3 \cdot \frac{(4n)!}{(4n+4)!} \cdot \left(\frac{n!}{(n+1)!}\right)^2$$

$(2n+2)! = (2n+2)(2n+1)\cdot(2n)!$, $(4n+4)! = (4n+4)(4n+3)(4n+2)(4n+1)\cdot(4n)!$, $(n+1)! = (n+1)\cdot n!$

$$= \lim_{n\to\infty} \frac{((2n+2)(2n+1))^3}{(4n+4)(4n+3)(4n+2)(4n+1)\cdot(n+1)^2} = \frac{(2\cdot 2)^3}{4^4} = \frac{1}{4}$$

Since $L = \frac{1}{4} < 1$, $\sum_{n=0}^{\infty} \frac{((2n)!)^3}{(4n)!(n!)^2}$ converges by R.T.