Do not forget to wri
your ***full name*** *and*
your ***Bilkent ID number****, and*
sign *on the upper right corner*
of your paper.

Final Exam Question 2.

Find all points $P(a,b,c)$ on the paraboloid $z = x^2 + y^2$ such that the tangent plane to the paraboloid at the point P contains the line:

$$L : x = 3t+1, \quad y = t-1, \quad z = 14t-3; \quad (-\infty < t < \infty)$$

Show all your work!
Explain your reasoning fully and in detail using correct mathematical notation and terminology, and in well-formed mathematical and English sentences!

Let $f(x,y) = x^2+y^2$. An equation for the tangent plane is:

$$z = f_x(a,b)\cdot(x-a) + f_y(a,b)\cdot(y-b) + f(a,b)$$

Since $f_x = 2x$ and $f_y = 2y$, this becomes:

$$z = 2a\cdot(x-a) + 2b\cdot(y-b) + a^2+b^2,$$

OR $$z = 2ax + 2by - a^2 - b^2.$$

The line L is contained in the tangent plane if and only if

$$14t - 3 = 2a\cdot(3t+1) + 2b\cdot(t-1) - a^2 - b^2 \quad \text{for all } t.$$

This is equivalent to:

$$14 = 6a + 2b \quad \Rightarrow \quad b = 7 - 3a$$

and

$$-3 = 2a - 2b - a^2 - b^2 \quad \Rightarrow \quad -3 = 2a - 2\cdot(7-3a) - a^2 - (7-3a)^2$$

$$\Downarrow$$

$$10a^2 - 50a + 60 = 0$$

$$\Downarrow$$

$$a = 2 \quad \text{or} \quad a = 3$$

$$b = 1 \qquad b = -2$$

Hence $(a,b,c) = (2,1,5)$ and $(3,-2,13)$ are the only points satisfying the condition.