3. The *combustion equation*

$$u_{xx} + u_{yy} = -e^u$$

arises in the study of self-propagating exothermic oxidative chemical reactions in thermochemistry. Find all possible values of the triple (a, b, c) of constants for which the function

$$u(x, y) = a \ln(bx^2 + by^2 + c)$$

satisfies the combustion equation for all (x, y) with $x^2 + y^2 < 1$ as well as the condition $u(x, y) = 0$ for all (x, y) with $x^2 + y^2 = 1$.

$$u_x = a \cdot \frac{1}{bx^2+by^2+c} \cdot 2bx$$

$$u_{xx} = a \cdot \frac{-1}{(bx^2+by^2+c)^2} \cdot (2bx)^2 + a \cdot \frac{1}{bx^2+by^2+c} \cdot 2b$$

Similarly:

$$u_{yy} = a \cdot \frac{-1}{(bx^2+by^2+c)^2} \cdot (2by)^2 + a \cdot \frac{1}{bx^2+by^2+c} \cdot 2b$$

$$u_{xx}+u_{yy} = \frac{-4ab^2(x^2+y^2) + 4ab(bx^2+by^2+c)}{(bx^2+by^2+c)^2} = \frac{4abc}{(bx^2+by^2+c)^2}$$

$$e^u = \exp(a \ln(bx^2+by^2+c)) = (bx^2+by^2+c)^a$$

Hence: $u_{xx}+u_{yy} = -e^u$ for all $x^2+y^2<1 \iff a=-2$ and $4abc=-1 \iff a=-2$ and $bc=\frac{1}{8}$

As $a \neq 0$, $u(x,y)=0$ for all $x^2+y^2=1 \iff a\ln(b+c)=0 \iff b+c=1$

$$\left.\begin{array}{l} bc=\frac{1}{8} \Rightarrow c=\frac{1}{8b} \\ b+c=1 \end{array}\right\} \Rightarrow b+\frac{1}{8b}=1 \Rightarrow b^2-b+\frac{1}{8}=0 \Rightarrow b=\frac{1}{2}\left(1\pm\frac{1}{\sqrt{2}}\right)$$

$$\Downarrow$$

$$c=\frac{1}{2}\left(1\mp\frac{1}{\sqrt{2}}\right)$$

Hence, $(a,b,c) = \left(-2, \frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right), \frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)\right)$ and $\left(-2, \frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right), \frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right)\right)$ are the only triples satisfying the conditions.