Do not forget to write your **full name** and your **Bilkent ID number**, and **sign** on the upper right corner of your paper.

Final Exam Question 2.

a. Find the first four nonzero terms of the Maclaurin series of

$$\cos\left(\frac{x}{x+1}\right)$$

using the Maclaurin series:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{for all } x \quad \text{and} \quad \frac{1}{x+1} \overset{\circledast}{=} 1 - x + x^2 - x^3 + \cdots \quad \text{for } |x| < 1$$

b. For each value of the constant p, determine whether the series

$$\sum_{n=1}^{\infty} n^p \left(\cos\left(\frac{1}{n+1}\right) - \cos\left(\frac{1}{n}\right) \right)$$

converges or diverges.

Show all your work!
Explain your reasoning fully and in detail using correct mathematical notation and terminology, and in well-formed mathematical and English sentences!

(a) For $|x|<1$ and up to the x^4 term:

$$\circledast \underset{d/dx}{\Rightarrow} \frac{-1}{(1+x)^2} = -1+2x-3x^2+4x^3-5x^4+\cdots \underset{\times(-x^2)}{\Rightarrow} \frac{x^2}{(1+x)^2} = x^2-2x^3+3x^4-\cdots$$

$$\Rightarrow \frac{x^4}{(1+x)^4} = \left(\frac{x^2}{1+x^2}\right)^2 = x^4+\cdots$$

Hence: $\cos\left(\frac{x}{x+1}\right) = 1 - \frac{1}{2!}\left(\frac{x}{x+1}\right)^2 + \frac{1}{4!}\left(\frac{x}{x+1}\right)^4 - \cdots = 1 - \frac{1}{2}(x^2-2x^3+3x^4-\cdots) + \frac{1}{24}(x^4+\cdots)+\cdots$

$$\Rightarrow \cos\left(\frac{x}{x+1}\right) = 1 - \frac{1}{2}x^2 + x^3 - \frac{35}{24}x^4 + \cdots \quad \text{for } |x|<1$$

(b) $\cos\left(\frac{x}{x+1}\right) - \cos x = \left(1 - \frac{1}{2}x^2 + x^3 - \frac{35}{24}x^4 + \cdots\right) - \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots\right) = x^3 - \frac{3}{2}x^4 + \cdots$ for $|x|<1$

$$\Rightarrow \cos\left(\frac{1}{n+1}\right) - \cos\left(\frac{1}{n}\right) = \frac{1}{n^3} - \frac{3}{2}\cdot\frac{1}{n^4} + \cdots \quad \text{for } n>1$$

Hence: $c = \lim_{n\to\infty} \dfrac{n^p \cdot \left(\cos\left(\frac{1}{n+1}\right) - \cos\left(\frac{1}{n}\right)\right)}{\frac{1}{n^{3-p}}} = \lim_{n\to\infty} \dfrac{n^p \cdot \left(\frac{1}{n^3} - \frac{3}{2}\frac{1}{n^4} + \cdots\right)}{\frac{1}{n^{3-p}}} = \lim_{n\to\infty}\left(1 - \frac{3}{2}\cdot\frac{1}{n} + \cdots\right) = 1$

As $0<c<\infty$ and $\sum \frac{1}{n^{3-p}}$ converges if $3-p>1$ and diverges if $3-p\le 1$,

$$\sum_{n=1}^{\infty} n^p \cdot \left(\cos\left(\frac{1}{n+1}\right) - \cos\left(\frac{1}{n}\right)\right) \begin{cases} \text{converges if } p<2 \\ \text{diverges if } p\geq 2 \end{cases}$$

by the Limit Comparison Test.