5. Suppose that the function $f(x)$ defined on the interval $(-R, R)$ by a power series

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

which has radius of convergence $R > 0$ satisfies $f(0) = 1$ and $f'(x) = f(x/2)$ for all x in $(-R, R)$.

a. Express the coefficients of the following series in terms of $c_0, c_1, c_2, \ldots, c_n, \ldots$ by filling in the boxes. No explanation is required.

$$f'(x) = \boxed{c_1} + \boxed{2c_2}\, x + \boxed{3c_3}\, x^2 + \cdots + \boxed{(n+1)c_{n+1}}\, x^n + \cdots$$

$$f(x/2) = \boxed{c_0} + \boxed{\tfrac{1}{2}c_1}\, x + \boxed{\tfrac{1}{4}c_2}\, x^2 + \cdots + \boxed{\tfrac{1}{2^n}c_n}\, x^n + \cdots$$

b. Give a recurrence relation for $\{c_n\}_{n=0}^{\infty}$ by filling in the boxes. No explanation is required.

$$c_0 = \boxed{1} \quad \text{and} \quad c_{n+1} = \boxed{\frac{1}{2^n \cdot (n+1)}} \cdot c_n \quad \text{for } n \geq 0$$

c. Find R.

$$\frac{1}{R} = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} = \lim_{n\to\infty} \frac{1}{2^n \cdot (n+1)} = 0 \Rightarrow R = \infty$$

d. Determine whether $f(-2)$ is positive or negative.

$$f(-2) = \sum_{n=0}^{\infty} c_n \cdot (-2)^n = \sum_{n=0}^{\infty} (-1)^n \cdot 2^n c_n = \sum_{n=0}^{\infty} (-1)^n b_n \quad \text{where } b_n = 2^n c_n > 0 \text{ for } n \geq 0.$$

By Part c, $\lim_{n\to\infty} b_n = 0$

By Part b, $b_{n+1} = \dfrac{1}{2^{n-1}\cdot(n+1)} b_n \leq b_n$ for $n \geq 1$ as $2^{n-1}\cdot(n+1) \geq 1 \cdot 1 = 1$ for $n \geq 1$

Therefore, by AST:

$$f(-2) \leq \sum_{n=0}^{4} (-1)^n b_n = 1 - 2 + 1 - \frac{1}{6} + \frac{1}{96} = -\frac{5}{32} < 0$$

$\Rightarrow f(-2)$ is negative.