4. Suppose that a differentiable function $f(x, y, z)$ satisfies the following conditions:

❶ $\dfrac{\partial f}{\partial y}(3, 1, 3) = 1.$

❷ The parametric curve

$$C_1 : \ \mathbf{r}_1 = (3 + 2t)\,\mathbf{i} + (1 - t^2)\,\mathbf{j} + (3 - 5t + t^2)\,\mathbf{k} \quad (-\infty < t < \infty)$$

is contained in the level surface S of $f(x, y, z)$ passing through the point $(3, 1, 3)$.

❸ The parametric curve

$$C_2 : \ \mathbf{r}_2 = (2 + t^2)\,\mathbf{i} + (2t^3 - 1)\,\mathbf{j} + (2t + 1)\,\mathbf{k} \quad (-\infty < t < \infty)$$

is also contained in the level surface S of $f(x, y, z)$ passing through the point $(3, 1, 3)$.

Find $\dfrac{\partial f}{\partial z}(3, 1, 3)$.

$$\vec{v}_1 = \frac{d\vec{r}_1}{dt} = 2\vec{i} - 2t\vec{j} + (-5+2t)\vec{k} \Rightarrow \vec{v}_1\Big|_{t=0} = 2\vec{i} - 5\vec{k}$$

$$\vec{v}_2 = \frac{d\vec{r}_2}{dt} = 2t\vec{i} + 6t^2\vec{j} + 2\vec{k} \Rightarrow \vec{v}_2\Big|_{t=1} = 2\vec{i} + 6\vec{j} + 2\vec{k}$$

Let $\vec{n} = \vec{v}_1\Big|_{t=0} \times \vec{v}_2\Big|_{t=1} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -5 \\ 2 & 6 & 2 \end{vmatrix} = 30\vec{i} - 14\vec{j} + 12\vec{k}$

$$\left.\begin{array}{l} ② \Rightarrow \vec{\nabla} f(3,1,3) \perp \vec{v}_1\big|_{t=0} \\ ③ \Rightarrow \vec{\nabla} f(3,1,3) \perp \vec{v}_2\big|_{t=1} \end{array}\right\} \Rightarrow \vec{\nabla} f(3,1,3) = c\vec{n}$$

for some scalar c

$$\Downarrow$$

$$\vec{\nabla} f(3,1,3) = 30c\vec{i} - 14c\vec{j} + 12c\vec{k}$$

$$① \Downarrow$$

$$-14c = 1$$

$$\Downarrow$$

$$c = -\frac{1}{14}$$

$$\frac{\partial f}{\partial z}(3,1,3) = 12c = 12 \cdot \left(-\frac{1}{14}\right) = -\frac{6}{7}$$