3. The *delayed heat equation*

$$u_t(x, t+1) = u_{xx}(x, t)$$

where $u(x,t)$ is the temperature as a function of the position x and the time t, arises in the problems of heat conduction in media which react to spatial variations in temperature with a time delay. For example, the temperature of meat as it is cooked can be modeled with the delayed heat equation.

Find all possible values of the pair of positive constants (a, b) for which the function

$$u(x,t) = \sin(ax - bt)$$

satisfies the delayed heat equation for all (x, t).

$u_x = a\cos(ax - bt)$

$u_{xx} = -a^2 \sin(ax - bt)$

$u_t = -b\cos(ax - bt)$

$$\left.\begin{array}{l} u_t(x,t+1) \overset{\circledast}{=} u_{xx}(x,t) \\ \text{for all } (x,t) \end{array}\right\} \iff \left\{\begin{array}{l} -b\cos(ax-bt-b) = -a^2\sin(ax-bt) \\ \text{for all } (x,t) \end{array}\right.$$

If $(x,t) = (0,0)$, then $\circledast \Rightarrow b\cos b = 0 \Rightarrow \cos b = 0$ as $b > 0$

$\Rightarrow b = \frac{\pi}{2} + 2\pi n$ or $\frac{3\pi}{2} + 2\pi n$, $n \geq 0$, n integer

If $b = \frac{\pi}{2} + 2\pi n$, $n \geq 0$, n integer, then:

$\cos(ax - bt - b) = \cos(ax - bt - \frac{\pi}{2} - 2\pi n) = \cos(ax - bt - \frac{\pi}{2}) = \sin(ax - bt)$

Hence, $\circledast$ holds for all $(x,t) \iff b = a^2 \iff a = b^{1/2}$ as $a > 0$

If $b = \frac{3\pi}{2} + 2\pi n$, $n \geq 0$, n integer, then:

$\cos(ax - bt - b) = \cos(ax - bt - \frac{3\pi}{2} - 2\pi n) = \cos(ax - bt - \frac{3\pi}{2}) = -\sin(ax - bt)$

Hence, $\circledast$ holds for all $(x,t) \iff b = -a^2$, which is impossible as $b > 0$.

Therefore, $(a,b) = \left(\left(\frac{\pi}{2} + 2\pi n\right)^{1/2}, \frac{\pi}{2} + 2\pi n\right)$, $n \geq 0$, n integer, are the only pairs of positive constants for which $u(x,t)$ satisfies the delayed heat equation.