5. Consider the power series $f(x) = \sum_{n=1}^{\infty} \left(\frac{1}{2} + \frac{1}{n}\right)^n x^n$.

a. Find the radius of convergence R of the power series.

$$\frac{1}{R} = \lim_{n\to\infty} |c_n|^{1/n} = \lim_{n\to\infty} \left| \left(\frac{1}{2}+\frac{1}{n}\right)^n \right|^{1/n} = \lim_{n\to\infty} \left(\frac{1}{2}+\frac{1}{n}\right) = \frac{1}{2} \Rightarrow R = 2$$

b. Determine whether $f(R)$ converges absolutely, converges conditionally, or diverges.

$f(2) = \sum_{n=1}^{\infty} \left(\frac{1}{2}+\frac{1}{n}\right)^n \cdot 2^n = \sum_{n=1}^{\infty} \left(1+\frac{2}{n}\right)^n$ diverges by nTT because

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1+\frac{2}{n}\right)^n = e^2 \neq 0$ by "useful limits".

c. Determine whether $f(-R)$ converges absolutely, converges conditionally, or diverges.

$f(-2) = \sum_{n=1}^{\infty} \left(\frac{1}{2}+\frac{1}{n}\right)^n \cdot (-2)^n = \sum_{n=1}^{\infty} (-1)^n \left(1+\frac{2}{n}\right)^n$ diverges by nTT because

$\{a_n\}_{n=1}^{\infty} = \left\{(-1)^n \left(1+\frac{2}{n}\right)^n\right\}_{n=1}^{\infty}$ diverges as $\lim_{n\to\infty} \left(1+\frac{2}{n}\right)^n = e^2 \neq 0$ by "useful limits".

d. Choose a suitable positive integer M and show that $f(1) < M$. ☞ You will earn $\max\{22 - 3M, 0\}$ points from this part for a completely correct solution.

$$f(1) = \sum_{n=1}^{\infty} \left(\frac{1}{2}+\frac{1}{n}\right)^n = \frac{3}{2} + 1 + \left(\frac{5}{6}\right)^3 + \left(\frac{3}{4}\right)^4 + \sum_{n=5}^{\infty} \left(\frac{1}{2}+\frac{1}{n}\right)^n$$

$$< \frac{3}{2} + 1 + \frac{125}{216} + \frac{81}{256} + \sum_{n=5}^{\infty} \left(\frac{7}{10}\right)^n < \frac{3}{2} + 1 + \frac{7}{12} + \frac{1}{3} + \frac{(7/10)^5}{1-\frac{7}{10}}$$

$$= \frac{41}{12} + \left(\frac{7}{10}\right)^5 \cdot \frac{10}{3} = \frac{41}{12} + \frac{7 \cdot 49 \cdot 49}{3 \cdot 100 \cdot 100} < \frac{41}{12} + \frac{7}{12} = 4$$