4. Determine whether each of the following series is convergent or divergent.

a. $\displaystyle\sum_{n=2}^{\infty}(-1)^n \frac{\ln n}{2^n}$

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|(-1)^{n+1}\frac{\ln(n+1)}{2^{n+1}}\right|}{\left|(-1)^n\cdot\frac{\ln n}{2^n}\right|} = \lim_{n\to\infty} \frac{1}{2}\cdot\frac{\ln(n+1)}{\ln n} = \frac{1}{2} < 1$$

$$\Rightarrow \sum_{n=2}^{\infty}(-1)^n\cdot\frac{\ln n}{2^n} \text{ converges by RT.}$$

$$\lim_{n\to\infty}\frac{\ln(n+1)}{\ln n} = \lim_{x\to\infty}\frac{\ln(x+1)}{\ln x} \overset{\text{L'H}}{=} \lim_{x\to\infty}\frac{\frac{1}{x+1}}{\frac{1}{x}} = \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = 1$$

b. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^{n-\ln n}}$

$$L = \lim_{n\to\infty}|a_n|^{1/n} = \lim_{n\to\infty}\left|\frac{1}{2^{n-\ln n}}\right|^{1/n} = \lim_{n\to\infty}\frac{1}{2^{1-\frac{\ln n}{n}}} = \frac{1}{2} < 1$$

$\frac{\ln n}{n} \to 0$ by "useful limits"

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{2^{n-\ln n}} \text{ converges by nRT.}$$

c. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{2^{n/\ln n}}$

Since $\displaystyle\lim_{n\to\infty}\frac{(\ln n)^2}{n} = 0$ by "useful limits", for all sufficiently large n:

$$0 < \frac{(\ln n)^2}{n} < \frac{\ln 2}{2} \Rightarrow 2\ln n < \frac{n}{\ln n}\cdot\ln 2 \Rightarrow n^2 < 2^{n/\ln n} \Rightarrow 0 < \frac{1}{2^{n/\ln n}} < \frac{1}{n^2}.$$

Hence $\displaystyle\sum_{n=2}^{\infty}\frac{1}{2^{n/\ln n}}$ converges by DCT as $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges (p-series with $p=2>1$).