3. A sequence $\{a_n\}_{n=1}^{\infty}$ satisfies the recursion relation

$$a_n = \frac{10}{9}\max\{a_{n-1}, a_{n-2}\} - \min\{a_{n-1}, a_{n-2}\}$$

for $n \geq 3$.

a. In this part suppose that $a_1 = 1$ and $a_2 = 1/9$.

❶ Fill in the following boxes with the correct values. No explanation is required.

$a_3 = \boxed{1}$ $\quad a_4 = \boxed{1}$ $\quad a_{2018} = \boxed{\frac{1}{9}}$

❷ Does the sequence $\{a_n\}_{n=1}^{\infty}$ converge or diverge? Carefully prove your claim.

$a_5 = \frac{10}{9}\max\{a_4, a_3\} - \min\{a_4, a_3\} = \frac{10}{9}\max\{1,1\} - \min\{1,1\} = \frac{10}{9} - 1 = \frac{1}{9}$

Since each term depends only on the previous two terms, the pattern $1, \frac{1}{9}, 1, 1, \frac{1}{9}, 1, 1, \frac{1}{9}, 1, \ldots$ repeats. Hence $\{a_n\}_{n=1}^{\infty}$ diverges.

b. In this part suppose that $a_1 = 1$ and $a_2 = 2/3$.

❶ Fill in the following boxes with the correct values. No explanation is required.

$a_3 = \boxed{\frac{4}{9}}$ $\quad a_4 = \boxed{\frac{8}{27}}$ $\quad a_{2018} = \boxed{\left(\frac{2}{3}\right)^{2017}}$

❷ Does the sequence $\{a_n\}_{n=1}^{\infty}$ converge or diverge? Carefully prove your claim.

$a_1 = 1 = \left(\frac{2}{3}\right)^{1-1}$, $a_2 = \frac{2}{3} = \left(\frac{2}{3}\right)^{2-1}$; and if $a_{n-2} = \left(\frac{2}{3}\right)^{n-3}$ and $a_{n-1} = \left(\frac{2}{3}\right)^{n-2}$, then:

$a_n = \frac{10}{9}\max\{a_{n-1}, a_{n-2}\} - \min\{a_{n-1}, a_{n-2}\} = \frac{10}{9}\max\left\{\left(\frac{2}{3}\right)^{n-2}, \left(\frac{2}{3}\right)^{n-3}\right\} - \min\left\{\left(\frac{2}{3}\right)^{n-2}, \left(\frac{2}{3}\right)^{n-3}\right\}$

$= \frac{10}{9}\cdot\left(\frac{2}{3}\right)^{n-3} - \left(\frac{2}{3}\right)^{n-2} = \left(\frac{2}{3}\right)^{n-3}\left(\frac{10}{9} - \frac{2}{3}\right) = \left(\frac{2}{3}\right)^{n-3}\cdot\frac{4}{9} = \left(\frac{2}{3}\right)^{n-1}.$

Hence $a_n = \left(\frac{2}{3}\right)^{n-1}$ for all $n \geq 1$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{2}{3}\right)^{n-1} = 0$. Hence $\{a_n\}_{n=1}^{\infty}$ converges.