1. Find all values of the constant a for which the limit

$$\lim_{x\to 0} \frac{xe^{ax^2} - \sin x}{x^5}$$

exists and evaluate the limit for each of these values of a.

$$\lim_{x\to 0} \frac{xe^{ax^2} - \sin x}{x^5} \overset{\text{l'H}}{=} \lim_{x\to 0} \frac{e^{ax^2} + x\cdot 2ax\, e^{ax^2} - \cos x}{5x^4}$$

$$\overset{\text{l'H}}{=} \lim_{x\to 0} \frac{2ax e^{ax^2} + 4ax e^{ax^2} + 2ax^2\cdot 2ax\, e^{ax^2} + \sin x}{20x^3}$$

$$\overset{\text{l'H}}{=} \lim_{x\to 0} \frac{6a e^{ax^2} + 6ax\cdot 2ax\, e^{ax^2} + 12a^2x^2 e^{ax^2} + 4a^2x^3\cdot 2ax\, e^{ax^2} + \cos x}{60x^2}$$

This limit does not exist unless $6a+1=0$. So, $a=-\frac{1}{6}$ from here on:

$$= \lim_{x\to 0} \frac{-e^{-x^2/6} + \frac{2}{3}x^2 e^{-x^2/6} - \frac{1}{27}x^4 e^{-x^2/6} + \cos x}{60x^2}$$

$$\overset{\text{l'H}}{=} \lim_{x\to 0} \frac{\frac{x}{3}e^{-x^2/6} + \frac{4}{3}x e^{-x^2/6} - \frac{2}{9}x^3 e^{-x^2/6} - \frac{4}{27}x^3 e^{-x^2/6} + \frac{1}{81}x^5 e^{-x^2/6} - \sin x}{120x}$$

$$= \frac{1}{120}\lim_{x\to 0}\left(\frac{5}{3}e^{-x^2/6} - \frac{10}{27}x^2 e^{-x^2/6} + \frac{1}{81}x^4 e^{-x^2/6} - \frac{\sin x}{x}\right)$$

$$= \frac{1}{120}\cdot\left(\frac{5}{3} - 0 + 0 - 1\right) = \frac{1}{180}$$