**1.** A differentiable function $f$ and the polynomials $p$, $q$, $r$, $s$ satisfy

$$(*)\qquad -x^2+\frac{9}{2}x-\frac{1}{2}=p(x)\le f(x)\le q(x)=x^2+2 \quad \text{for all } x<1\,,$$

and

$$(**)\qquad -\frac{3}{2}x^3+6x^2-6x+\frac{9}{2}=r(x)\le f(x)\le s(x)=-x^3+5x^2-5x+4 \quad \text{for all } x>1\,.$$

**a.** Find $f(1)$.

$$\left.\begin{array}{l}\lim\limits_{x\to 1^-} p(x)=-1^2+\frac{9}{2}\cdot 1-\frac{1}{2}=3\\ \lim\limits_{x\to 1^-} q(x)=1^2+2=3\end{array}\right\}\Rightarrow \lim_{x\to 1^-} f(x)=3 \text{ by } (*) \text{ and the Squeeze Theorem.}$$

$f$ is differentiable $\Rightarrow$ $f$ is continuous $\Rightarrow f(1)=\lim\limits_{x\to 1^-} f(x)=3$.

**b.** Find $f'(1)$.

For $h<0$:

$(*)\Rightarrow f(1+h)\le q(1+h)\Rightarrow f(1+h)-f(1)\le q(1+h)-q(1)\Rightarrow \dfrac{f(1+h)-f(1)}{h}\ge\dfrac{q(1+h)-q(1)}{h}$

$\Rightarrow f'(1)=\lim\limits_{h\to 0^-}\dfrac{f(1+h)-f(1)}{h}\ge\lim\limits_{h\to 0^-}\dfrac{q(1+h)-q(1)}{h}=q'(1)=2x\Big|_{x=1}=2$

For $h>0$:

$(**)\Rightarrow f(1+h)\le s(1+h)\Rightarrow f(1+h)-f(1)\le s(1+h)-s(1)\Rightarrow\dfrac{f(1+h)-f(1)}{h}\le\dfrac{s(1+h)-s(1)}{h}$

$\Rightarrow f'(1)=\lim\limits_{h\to 0^+}\dfrac{f(1+h)-f(1)}{h}\le\lim\limits_{h\to 0^+}\dfrac{s(1+h)-s(1)}{h}=s'(1)=(-3x^2+10x-5)\Big|_{x=1}=2$

So: $f'(1)\ge 2$ and $f'(1)\le 2\Rightarrow f'(1)=2$

**c.** Show that there is a point $c$ such that $f'(c)=0$.

$$(**)\Rightarrow\begin{cases} f(2)\ge r(2)=-\frac{3}{2}\cdot 2^3+6\cdot 2^2-6\cdot 2+\frac{9}{2}=\frac{9}{2}>3\\ f(4)\le s(4)=-4^3+5\cdot 4^2-5\cdot 4+4=0<3\end{cases}$$

$f$ is differentiable $\Rightarrow f$ is continuous on $[2,4]$

$\Rightarrow$ By IVT, there is $c_0$ in $(2,4)$ such that $f(c_0)=3$.

Since $f$ is continuous on $[1,c_0]$ and differentiable on $(1,c_0)$,

by Rolle's Theorem, there is $c$ in $(1,c_0)$ such that $f'(c)=0$.