2. Suppose f is a twice-differentiable function on $(-\infty, \infty)$ satisfying the following conditions:

❶ $x = 4$ and $x = 11$ are the only critical points of f in the interval $(0, 15)$.

❷ $f(0) = 1,\ f(4) = -2,\ f(11) = 4,\ f(15) = -1$.

❸ $f'(0) = -1,\ f'(15) = -2$.

❹ $|f''(x)| \leq 1$ for all x in the interval $[0, 15]$.

a. Let $g(x) = 3f(x) - (f'(x))^2$. Show that the information given above is sufficient to determine the absolute maximum and minimum values of the function g on the interval $[0, 15]$, and find them.

$$g'(x) = 3f'(x) - 2f'(x)f''(x) = f'(x)\cdot(3 - 2f''(x))$$

$$g'(x) = 0 \Rightarrow f'(x) = 0 \quad \text{or} \quad f''(x) = \frac{3}{2}.$$

$f'(x) = 0 \Downarrow x = 4,\ x = 11$ for $0 < x < 15$

This equation has no solution in $(0, 15)$ as $f''(x) \leq 1$ by 4.

Critical points:

$x = 4 \Rightarrow g(4) = 3f(4) - f'(4)^2 = 3\cdot(-2) - 0^2 = -6$

$x = 11 \Rightarrow g(11) = 3f(11) - f'(11)^2 = 3\cdot 4 - 0^2 = 12$

Endpoints:

$x = 0 \Rightarrow g(0) = 3f(0) - f'(0)^2 = 3\cdot 1 - (-1)^2 = 2$

$x = 15 \Rightarrow g(15) = 3f(15) - f'(15)^2 = 3\cdot(-1) - (-2)^2 = -7$

Abs max and min of g on $[0, 15]$ are 12 and -7.

b. Show that there is a point c in the interval $(0, 15)$ such that $f''(c) = -1/2$.

f is twice-differentiable $\Rightarrow$ f' is differentiable and continuous everywhere.

$x = 11$ is a critical point of f and f' is defined at $x = 11$ $\Rightarrow f'(11) = 0$.

Applying MVT to f' on $[11, 15]$, we conclude that there is a point c in $(11, 15)$ such that:

$$f''(c) = \frac{f'(15) - f'(11)}{15 - 11} = \frac{-2 - 0}{4} = -\frac{1}{2}$$