2. Consider the function $f(x) = \dfrac{\sin(\pi x)}{\sqrt{x^2+x^3} - x}$.

a. Find the domain of f.

$\circledast$ $\sqrt{x^2+x^3} - x \neq 0 \iff \sqrt{x^2+x^3} \neq x \iff x^2+x^3 \neq x^2$ and $x \geq 0 \iff x \neq 0$

$\circledast$ $x^2+x^3 \geq 0 \iff x^2(1+x) \geq 0 \iff 1+x \geq 0 \iff x \geq -1$

$\Rightarrow$ Domain of f is $[-1,0) \cup (0,\infty)$.

b. Compute $f(5/4)$.

$$f\left(\frac{5}{4}\right) = \frac{\sin\left(\frac{5\pi}{4}\right)}{\sqrt{\left(\frac{5}{4}\right)^2+\left(\frac{5}{4}\right)^3} - \frac{5}{4}} = \frac{-\frac{1}{\sqrt{2}}}{\frac{5}{4}\cdot\left(\sqrt{\frac{9}{4}}-1\right)} = -\frac{4\sqrt{2}}{5}$$

c. Show that there is a positive real number x such that $f(x) = -1$.

$f\left(\frac{5}{4}\right) = -\sqrt{\frac{32}{25}} < -1 < 0 = f(1)$

Since f is continuous on $\left[1, \frac{5}{4}\right]$, by IVT there is a c in $\left(1, \frac{5}{4}\right)$ such that $f(c) = -1$.

d. Evaluate $\lim\limits_{x\to 0^-} f(x)$. [Do not use L'Hôpital's Rule!]

$$\lim_{x\to0^-} f(x) = \lim_{x\to0^-} \frac{\sin\pi x}{\sqrt{x^2+x^3} - x} = \lim_{x\to0^-} \frac{\sin\pi x}{|x|\sqrt{1+x} - x} = \lim_{x\to0^-} \frac{\sin\pi x}{-x\sqrt{1+x} - x}$$

$$= -\pi \cdot \lim_{x\to0^-} \frac{\sin\pi x}{\pi x} \cdot \lim_{x\to0^-} \frac{1}{\sqrt{1+x}+1} = -\pi\cdot 1\cdot\frac{1}{2} = -\frac{\pi}{2}$$

e. Evaluate $\lim\limits_{x\to 0^+} xf(x)$. [Do not use L'Hôpital's Rule!]

$$\lim_{x\to0^+} xf(x) = \lim_{x\to0^+} \frac{x\sin\pi x}{\sqrt{x^2+x^3} - x} = \lim_{x\to0^+} \frac{x\sin\pi x}{|x|\sqrt{1+x} - x} = \lim_{x\to0^+} \frac{x\sin\pi x}{x\sqrt{1+x} - x}$$

$$= \lim_{x\to0^+} \frac{\sin\pi x}{\sqrt{1+x}-1} = \lim_{x\to0^+} \frac{\sin\pi x\cdot(\sqrt{1+x}+1)}{(\sqrt{1+x}-1)\cdot(\sqrt{1+x}+1)} = \pi\cdot\lim_{x\to0^+}\frac{\sin\pi x}{\pi x}\cdot\lim_{x\to0^+}(\sqrt{1+x}+1)$$

$$= \pi\cdot 1\cdot 2 = 2\pi$$